

Topology of partition of measures by fans and the second obstruction

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Abstract

The simultaneous partition problems are classical problems of the combinatorial geometry which have the natural flavor of the equivariant topology. The k -fan partition problems have attracted a lot of attention [1], [2], [3] and forced some hard concrete combinatorial calculations in the equivariant cohomology [4]. These problems can be reduced, by a beautiful scheme of [2], to a "typical" question of the existence of a \mathbb{D}_{2n} equivariant map $f : V_2(\mathbb{R}^3) \rightarrow W_n - \cup \mathcal{A}(\alpha)$, where $V_2(\mathbb{R}^3)$ is the space of all orthonormal 2-frames in \mathbb{R}^3 and $W_n - \cup \mathcal{A}(\alpha)$ is the complement of the appropriate arrangement. We introduce the *target extension scheme* which allow us to use the equivariant obstruction theory as a tool for proving that: for every two proper measures on the sphere S^2 , and any $\alpha = (a, a + b, b) \in \mathbb{R}_{>0}^3$, there exists an α -partition of theses measures by a 3-fan.

The significance of these results, among other, is that, beside negative results [4], the equivariant obstruction theory can pull off some positive results, which were not attained by other means.

1 Introduction

1.1 Problem

A k -fan $(x; l_1, l_2, \dots, l_k)$ on the sphere S^2 is formed of a point x , called the center of the fan, and k great semicircles l_1, \dots, l_k emanating from x . We always assume counter clockwise enumeration on great semicircles l_1, \dots, l_k of a k -fan. Sometimes instead of lines we use open angular sectors σ_i between l_i and l_{i+1} , $i = 1, \dots, k$. In that case we denote a k -fan with $(x; \sigma_1, \sigma_2, \dots, \sigma_k)$.

Let $\mu_1, \mu_2, \dots, \mu_m$ be *proper* Borel probability measures on S^2 . Measure μ is *proper* if $\mu([a, b]) = 0$ for any circular arc $[a, b] \subset S^2$ and $\mu(U) > 0$ for each nonempty open set $U \subset S^2$. All results can be extended to more general measures, including the counting measures of finite sets, see [2] for related examples.

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Let $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathbb{R}_{\geq 0}^k$ be a vector where $\alpha_1 + \alpha_2 + \dots + \alpha_k = 1$. The general problem stated in [2] is:

Problem 1 Find all triples $(m, k, \alpha) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}^k$ such that for any collection of m measures $\{\mu_1, \mu_2, \dots, \mu_m\}$, there exists a k -fan $(x; l_1, \dots, l_k)$ with the property

$$(\forall i = 1, \dots, k) (\forall j = 1, \dots, m) \mu_j(\sigma_i) = \alpha_i$$

That kind of a k -fan $(x; l_1, \dots, l_k)$ is called an α -partition for the collection of measures $\{\mu_j\}_{j=1}^m$.

The analysis given in [2] shows that the most interesting triples are $(3, 2, \alpha)$, $(2, 3, \alpha)$, $(2, 4, \alpha)$. Known results can be summed in the following table:

m/k	2	3	4
2	$R_{\geq 0}^2, [2]$	$\begin{pmatrix} \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \\ \frac{*}{5}, \frac{*}{5}, \frac{*}{5} \end{pmatrix} [2]$	$\begin{pmatrix} \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \\ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \end{pmatrix} [2] [3]$ Topology can't help any more [4].
3	$\begin{pmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{3}, \frac{2}{3} \\ \frac{2}{3}, \frac{1}{3} \end{pmatrix} [2]$	\emptyset	\emptyset

In this paper we try to fill this table a little bit more.

1.2 The statement of results

We are interested in the problem of 3-fan partitions of two measures on S^2 , respectively. We prove the following result:

Theorem 2 Lets choose $\alpha = (a, a+b, b) \in \mathbb{R}_{\geq 0}^3$ such that $2a+2b=1$. Then any two proper measures μ and ν on the sphere S^2 admit an α -partition by a 3-fan $\mathbf{p} = (x; l_1, l_2, l_3)$, i. e.

$$\mathcal{A}_{2,3} \supseteq \{(a, a+b, b) \in \mathbb{R}^3 \mid a, b > 0, 2a+b=1\}.$$

Remark 3 The case $a=b=1$ was already considered in [2]

1.3 The solution scheme

The solution of the problem has two natural parts. The first part is the reduction of the problem to the question of the existence of the appropriate equivariant map. The second part is and topological effort to disprove the existence of a such map. There is also a third part of the proof, the limit argument. It extends the result from rational triples to real triples, but we omitted it because it is the standard part of every similiar proof.

Reduction to the equivariant problem:

- The configuration space / test map procedure of Imre Bárány and Jiří Matoušek from [2] reduces the problem to the question: Is there an $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n})$, such that there is no \mathbb{D}_{2n} -map $V_2(\mathbb{R}^3) \rightarrow W_n \setminus \cup \mathcal{A}(\alpha)$?

- The extension of scalars equivalence from homological algebra, [5], allows us to change the initial equivariant question to: Is there an $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n})$, such that there is no \mathbb{Q}_{4n} -map $S^3 \rightarrow W_n \setminus \cup \mathcal{A}(\alpha)$?
- The elementary obstruction theory can convince us that the actual free \mathbb{Q}_{4n} action on S^3 is not of the essential importance. Thus we can always change it the way it pleases us.

Obstruction theory approach:

- The dimensional reasons of the problem, $\pi_1(W_n \setminus \cup \mathcal{A}(\alpha)) \neq \emptyset$, implies that there are two essential obstructions. This forces us to introduce the target extension scheme, and instead of dealing with two obstructions we have only one to compute. The target extension scheme extends the \mathbb{Q}_{4n} space $W_n \setminus \cup \mathcal{A}(\alpha)$ in such a way that connectivity increases by one. Thus, the problem of the existence of the \mathbb{Q}_{4n} -map from S^3 to this extension has only one obstruction.
- Once we extend the complement $W_n \setminus \cup \mathcal{A}(\alpha)$ we use the classical obstruction theory and the "map in general position" method to prove that appropriate obstruction is not zero. This part of the proof goes in a small number of steps, like we described in the section 3.7.
- The main step is the identification of the obstruction cocycle. This is the point where many previous papers have been broken ([12], proof of theorem 4.2, equality (25) and [18], proof of theorem 6.1). We use the clear geometrical picture and simple testing methods (in sections 4.8 and 4.9) to interpret the obstruction element as a non-zero element of the appropriate coinvariant group.

This scheme of the solution, with different extensions, can be applied to other cases of the 3-fan / 2-measures problem, as well on the 2-fan / 3-measures problem. The main idea of extending the target space, of course with some variations, can be applied on every problem of the existence of the equivariant map to a complement of an arrangement, or to any space that has natural extension candidates.

2 From a partition problem to an equivariant problem

2.1 Reduction to the equivariant problem

The reduction of the fan partition problem to the equivariant problem is done by the configurations space / test map scheme. The main idea is to look at the space of all possible solutions and to rephrase the question in terms of coincidences of the associated test map. Imre Bárány and Jiří Matoušek demonstrated in [2] that the test map scheme can be applied on the problem of α -partitioning of m -measures on S^2 by spherical k -fans. In a very elegant way this problem was reduced to the problem of the existence of the appropriate equivariant map. We briefly review this reduction for the $(3, 2)$ case of this problem.

The configuration space. Let μ and ν be two proper Borel probability measures on S^2 , and F_k the space of all k -fans on the sphere S^2 . The space X_μ of all possible solutions associated to the measure μ is defined by

$$X_\mu = \{(x; l_1, \dots, l_n) \in F_n \mid (\forall i = 1, \dots, n) \mu(\sigma_i) = \frac{1}{n}\}.$$

Observe that every n -fan $(x; l_1, \dots, l_n) \in X_\mu$ is completely determined by the pair (x, l_1) or equivalently, the pair (x, y) , where y is the unit tangent vector to l_1 at x . Thus, the space X_μ is a Stiefel manifold $V_2(\mathbb{R}^3)$ of all orthonormal 2-frames in \mathbb{R}^3 . Keep in mind that $V_2(\mathbb{R}^3) \cong SO(3) \cong \mathbb{R}P^3$.

Test map. Let \mathbb{R}^n be an Euclidean space with the standard orthonormal basis e_1, e_2, \dots, e_n and the associated coordinate functions x_1, x_2, \dots, x_n . Let W_n be the hyperplane $\{x \in \mathbb{R}^n \mid x_1 + x_2 + \dots + x_n = 0\}$ in \mathbb{R}^n , and suppose that α -vectors have the following form

$$\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}) \in \frac{1}{n} \mathbb{N}^3 \subseteq \mathbb{Q}^3,$$

where $a_1 + a_2 + a_3 = n$. Then test maps for the $(3, 2)$ fan problem are defined by

$$F_\nu : X_\mu \rightarrow W_n \quad F_\nu(\mathbf{p}) = (\nu(\sigma_1) - \frac{1}{n}, \dots, \nu(\sigma_n) - \frac{1}{n})$$

The action. The dihedral group $\mathbb{D}_{2n} = \langle j, \varepsilon \mid \varepsilon^n = j^2 = 1, \varepsilon j = j \varepsilon^{n-1} \rangle$ acts both on the possible solution space X_μ and the linear subspace $W_n \subseteq \mathbb{R}^n$ by

$$X_\mu : \begin{cases} \varepsilon(x; l_1, \dots, l_n) = (x; l_n, l_1, \dots, l_{n-1}) \\ j(x; l_1, \dots, l_n) = (-x; l_1, l_n, l_{n-1}, \dots, l_2) \end{cases}, \quad W_n : \begin{cases} \varepsilon(x_1, \dots, x_n) = (x_2, \dots, x_n, x_1) \\ j(x_1, \dots, x_n) = (x_n, \dots, x_2, x_1) \end{cases},$$

for $(x; l_1, \dots, l_n) \in X_\mu$ and $(x_1, \dots, x_n) \in W_n$. The action of \mathbb{D}_{2n} on X_μ is **free**.

Observe that the space of possible solutions X_μ is \mathbb{D}_{2n} -homeomorphic to the manifold $V_2(\mathbb{R}^3)$, where $V_2(\mathbb{R}^3)$ is a \mathbb{D}_{2n} -space given by

$$\varepsilon(x, y) = (x, R_x(\frac{2\pi}{n})(y)), \quad j(x, y) = (-x, y),$$

and $R_x(\theta) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the rotation round the axes determined by x through the angle θ .

The test space. The test space in this problem is the union $\cup \mathcal{A}(\alpha) \subset W_n$ of a smallest \mathbb{D}_{2n} -invariant linear subspace arrangement $\mathcal{A}(\alpha)$, which contains linear subspace $L(\alpha) \subset W_n$. The subspace $L(\alpha)$ is defined by

$$L(\alpha) = \{x \in \mathbb{R}^n \mid \xi_1(x) = \xi_2(x) = \xi_3(x) = 0\} \subseteq W_n,$$

where

$$\xi_1(x) = x_1 + \dots + x_{a_1}, \quad \xi_2(x) = x_{a_1+1} + \dots + x_{a_1+a_2}, \quad \xi_3(x) = x_{a_1+a_2+1} + \dots + x_n.$$

Since the test map F_ν is obviously \mathbb{D}_{2n} -equivariant, the following standard proposition is proved.

Proposition 4 *Let $\alpha = (\frac{a_1}{n}, \frac{a_2}{n}, \frac{a_3}{n}) \in \frac{1}{n} \mathbb{N}^3 \subseteq \mathbb{Q}^3$ be a vector such that $a_1 + a_2 + a_3 = n$. If there is no \mathbb{D}_{2n} -equivariant map*

$$F : V_2(\mathbb{R}^3) \rightarrow W_n \setminus \cup \mathcal{A}(\alpha)$$

then for any two measures μ and ν on S^2 , there exists an α -partition $(x; l_1, l_2, l_3)$ of measures μ and ν .

2.2 Modifying Problem

Knowing the fact that for an odd n , there always exists a \mathbb{Z}_n -map $f : S^3 \rightarrow V_2(\mathbb{R}^3)$, Bárány and Matoušek in [2] questioned if there is a \mathbb{Z}_n -map from the sphere S^3 to the complements of appropriate arrangements. To do something similar we extend the group, like in [18] and [4]. We use well known “extension of scalars” equivalence from homological algebra, [5] Section III.3.

The generalized quaternion group. Let $S^3 = S(\mathbb{H}) = Sp(1)$ be the group of all unit quaternions and let $\epsilon = \epsilon_{2n} = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \in S(\mathbb{H})$ be a root of unity. Group $\langle \epsilon \rangle$ is a subgroup of $S(\mathbb{H})$ of the order $2n$. Then, the generalized quaternion group, [6] p. 253, is the subgroup

$$\mathbb{Q}_{4n} = \{1, \epsilon, \dots, \epsilon^{2n-1}, j, \epsilon j, \dots, \epsilon^{2n-1} j\}$$

of S^3 , of the order $4n$. Let $H = \{1, \epsilon^n\} = \{1, -1\} \subset \mathbb{Q}_{4n}$. Then, it is not hard to see that the quotient group \mathbb{Q}_{4n}/H is isomorphic to the dihedral group \mathbb{D}_{2n} of the order $2n$.

Proposition 5 *Let the generalized quaternion group \mathbb{Q}_{4n} act on S^3 as a subgroup, and on W_n via already defined \mathbb{D}_{2n} action by the quotient homomorphism $\mathbb{Q}_{4n} \rightarrow \mathbb{Q}_{4n}/\{1, -1\} \cong \mathbb{D}_{2n}$. Then the following maps coexist:*

$$\mathbb{D}_{2n}\text{-map } V_2(\mathbb{R}^3) \rightarrow W_n \setminus \cup \mathcal{A}(\alpha) \text{ and } \mathbb{Q}_{4n}\text{-map } S^3 \rightarrow W_n \setminus \cup \mathcal{A}(\alpha).$$

By the coexistence we mean that the one map exists if and only if the other map exists, i.e. the one can't exist without the other.

Proof. Let us denote the target space $W_n \setminus \cup \mathcal{A}(\alpha)$ with T . Also, observe that $S^3/\{1, -1\} \cong \mathbb{RP}^3 \cong SO(3) \cong V_2(\mathbb{R}^3)$

\Rightarrow : Let $F : V_2(\mathbb{R}^3) \rightarrow T$ be a \mathbb{D}_{2n} -map. The quotient map $p : S^3 \rightarrow S^3/\{1, -1\} \cong V_2(\mathbb{R}^3)$ is a \mathbb{Q}_{4n} -map where the \mathbb{Q}_{4n} -acts on $V_2(\mathbb{R}^3)$ by the quotient homomorphism $\mathbb{Q}_{4n} \rightarrow \mathbb{Q}_{4n}/\{1, -1\} \cong \mathbb{D}_{2n}$. Since, \mathbb{Q}_{4n} acts on both $V_2(\mathbb{R}^3)$ and T via the quotient homomorphism, the given \mathbb{D}_{2n} -map $F : V_2(\mathbb{R}^3) \rightarrow T$ can also be seen as the \mathbb{Q}_{4n} -map. Thus, the composition

$$F \circ p : S^3 \longrightarrow S^3/\{1, -1\} \cong V_2(\mathbb{R}^3) \longrightarrow T$$

is the required \mathbb{Q}_{4n} -map $S^3 \rightarrow T$.

\Rightarrow : Let $G : S^3 \rightarrow T$ be a \mathbb{Q}_{4n} -map. Observe that $S^3/\{1, -1\} \cong V_2(\mathbb{R}^3)$ can be seen as a $\mathbb{D}_{2n} \cong \mathbb{Q}_{4n}/\{1, -1\}$ space by $\{1, -1\}x \xrightarrow{g\{1, -1\}} \{1, -1\}(gx)$, where $x \in S^3$ and $g \in \mathbb{Q}_{4n}$. Since the subgroup $\{1, -1\}$ acts trivially on T , there is a factorization of the map G through the quotient $h : S^3/\{1, -1\} \rightarrow T$ such that $G = h \circ p$. The map h is the required \mathbb{D}_{2n} -map

$$h(g\{1, -1\} \cdot \{1, -1\}x) = G(g \cdot x) = g \cdot G(x) = g \cdot h(\{1, -1\}x) = (g\{1, -1\}) \cdot h(\{1, -1\}x).$$

■

Remark 6 *The \mathbb{Q}_{4n} action on S^3 is **free**. Also, the \mathbb{Q}_{4n} action on W_n is the restriction of the following \mathbb{Q}_{4n} action on \mathbb{R}^n . Let e_1, \dots, e_n be the standard orthonormal basis in \mathbb{R}^n . The action is defined by*

$$\epsilon \cdot e_i = e_{i \bmod n+1} \text{ and } j \cdot e_i = e_{n-i+1}.$$

The free action on S^3 . Since the sphere S^3 is 2-connected it turns out that the particular \mathbb{Q}_{4n} -action on S^3 is not something we have to live with. The elementary equivariant obstruction theory allows us to prove the following useful fact.

Proposition 7 *If γ_1 and γ_2 are G -actions on S^3 and γ_1 is free, then there exists a G -map $f : S^3 \rightarrow S^3$ such that*

$$(\forall g \in G) (\forall x \in S^3) f(g \cdot_1 x) = g \cdot_2 f(x).$$

Proof. The statement is true because S^3 is 2-connected, γ_1 is free and so there are no obstructions to extend a G -map from 0-skeleton to S^3 . ■

Thus, when the time comes we will be able to use any free \mathbb{Q}_{4n} action on S^3 and we will have the favorite one.

3 Obstruction theory approach

Once again, denote the target space $W_n \setminus \cup \mathcal{A}(\alpha)$ with T . To answer a question of the existence of \mathbb{Q}_{4n} -map $S^3 \rightarrow T$ we will try to employ the classical obstruction theory. Since the maximal elements of the arrangement $\mathcal{A}(\alpha)$ are of the codimension $2 = (n-1) - (n-3)$ in W_n , the complement is connected and the first obstruction lives in $H_{\mathbb{Q}_{4n}}^2(S^3, \pi_1(T))$. In this case it is very hard even to identify this group, not to mention to identify the particular element in it. Even if we do manage to identify and calculate the first obstruction, there is a good chance that it is zero, so the second obstructions should be calculated. To omit this difficulties we use the nature of the target spaces, introduce the target extension scheme and then use the equivariant obstruction theory.

3.1 The Target extension scheme

According to the proposition 4 we would prefer to prove that there are no equivariant \mathbb{Q}_{4n} -maps $S^3 \rightarrow T$. Thus the following scheme can be of some help.

The basic idea of the target extension scheme is to *find a \mathbb{Q}_{4n} space E which contains the target space T and to prove that there is no \mathbb{Q}_{4n} -map $S^3 \rightarrow E$* . Thus, this would imply that there is no \mathbb{Q}_{4n} -map $S^3 \rightarrow T$.

Since the target space T is the complement of the arrangement, the basic idea can be refined as follows:

(A) *Increase the codimension of the arrangement:* Take an arbitrary hyper arrangement \mathcal{J} in W_n . By the *hyper* arrangement we mean the arrangement of hyperplanes and / or closed hyperplane halfspaces. Then form the minimal \mathbb{Q}_{4n} -invariant arrangement $\mathcal{A}(\mathcal{J}, \alpha)$ containing the family $\mathcal{J} \cap L(\alpha) = \{J \cap L(\alpha) \mid J \in \mathcal{J}\}$. Then inclusion $\cup \mathcal{A}(\mathcal{J}, \alpha) \subseteq \cup \mathcal{A}(\alpha)$ implies that $W_n - \cup \mathcal{A}(\mathcal{J}, \alpha) \supseteq W_n - \cup \mathcal{A}(\alpha)$. Observe that the dimension of maximal elements of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$ is $n-4$. Let us denote (when it suits us) the new union $\cup \mathcal{A}(\mathcal{J}, \alpha)$ by U^* , and the new complement $W_n - \cup \mathcal{A}(\mathcal{J}, \alpha)$ by T^* .

(B) *Apply the obstruction theory to the new question:* Is there a \mathbb{Q}_{4n} -map $S^3 \rightarrow T^*$. Since the codimension of maximal elements of the new defined arrangement in W_n is 3, the target space T^* is 1-connected and consequently 2-simple in the sense that $\pi_1(T^*)$ acts

trivially on $\pi_2(T^*)$. Thus by Hurewicz theorem $\pi_2(T^*) \cong [S^2, T^*] \cong H_2(T^*; \mathbb{Z})$. Then the part of the obstruction exact sequence ([8], [17]) we are interested in is

$$[S^3, T^*]_{\mathbb{Q}_{4n}} \xrightarrow{\theta} \text{Im} \left\{ [S^3_{(2)}, T^*]_{\mathbb{Q}_{4n}} \longrightarrow [S^3_{(1)}, T^*]_{\mathbb{Q}_{4n}} \right\} \xrightarrow{\tau_{\mathbb{Q}_{4n}}} H^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z}))$$

where $S^3_{(1)}$ and $S^3_{(2)}$ are respectively the 1- and 2-skeleton of $S^3 = P_{2n} * P_{2n}$ and $H_2(T^*; \mathbb{Z})$ is viewed as a \mathbb{Q}_{4n} -module. Since $[S^3_{(1)}, T^*]_{\mathbb{Q}_{4n}} = \{*\}$ is a one-element set and $[S^3_{(2)}, T^*]_{\mathbb{Q}_{4n}} \neq \emptyset$, the sequence becomes

$$[S^3, T^*]_{\mathbb{Q}_{4n}} \longrightarrow \{*\} \xrightarrow{\tau_{\mathbb{Q}_{4n}}} H^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z})).$$

The exactness means that the set $[S^3, T^*]_{\mathbb{Q}_{4n}} \neq \emptyset$ if and only if $\tau_{\mathbb{Q}_{4n}}(*) \in H^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z}))$ is equal to zero. The element $\tau_{\mathbb{Q}_{4n}}(*)$ depends only on T^* . Thus, the main question transforms in

3.2 A map in the general position

We evaluate the class $\tau_{\mathbb{Q}_{4n}}(*)$ by the so called "map in the general position" standard procedure, [17]. Since the set $[S^3_{(2)}, T^*]_{\mathbb{Q}_{4n}}$ is a point, it suffices to calculate an obstruction cocycle for the particular map h .

Let $h : S^3 \rightarrow W_n$ be an arbitrary \mathbb{Q}_{4n} -simplicial map which is in the general position. What we mean by the general position is that for any simplex σ in S^3

$$h(\sigma) \cap U^* \neq \emptyset \Rightarrow \dim(h(\sigma)) = 3 \text{ and } h(\sigma) \cap U^* = \{p_1, \dots, p_k\} \subset \text{inh}(\sigma).$$

Now let $h : S^3 \rightarrow W_n$ be a \mathbb{Q}_{4n} -map in the general position. Then the associated cohomology class of the obstruction cocycle $c_{\mathbb{Q}_{4n}}(h) \in C^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z})) = \text{Hom}_{\mathbb{Q}_{4n}}(C_3(S^3), H_2(T^*; \mathbb{Z}))$ for the map h is equal to $\tau(*)$, i.e. $[c_{\mathbb{Q}_{4n}}(h)] = \tau(*)$. In order to calculate it let us describe the cocycle $c_{\mathbb{Q}_{4n}}(h)$ a little bit closer. Let σ be an oriented 3-simplex in S^3 . Then $c_{\mathbb{Q}_{4n}}(h)(\sigma) \in H_2(T^*; \mathbb{Z})$ is the h_* image of the fundamental class of $\partial(\sigma) \cong S^2$ by the map $h_* : H_2(\partial(\sigma); \mathbb{Z}) \rightarrow H_2(T^*; \mathbb{Z})$, i.e.

$$c_{\mathbb{Q}_{4n}}(h)(\sigma) = h_*[\partial(\sigma)].$$

3.3 The nature of the obstruction cocycle

The following proposition will narrow our attention to the torsion part of the group $H^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z}))$ and will be the perfect control factor in our calculations. It can be also found in [4].

Proposition 8 *The cohomology class of the obstruction cocycle $c_{\mathbb{Q}_{4n}}(h)$ is a torsion element of the group $H^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z}))$.*

Proof. Let H be a subgroup of \mathbb{Q}_{4n} . The restriction map

$$r : H^3_{\mathbb{Q}_{4n}}(S^3, H_2(T^*; \mathbb{Z})) \rightarrow H^3_H(S^3, H_2(T^*; \mathbb{Z}))$$

on the cochain level sends $c \in C_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z}))$ a \mathbb{Q}_{4n} -cochain to now H -cochain $c \in C_H^3(S^3, H_2(T^*; \mathbb{Z}))$. The definition of the obstruction cocycle implies that $r(c_{\mathbb{Q}_{4n}}(h))$ is the obstruction cocycle for the extension of the H -map in the general position h . It is a known fact ([5] Section III.9. Proposition 9.5.(ii)) that the composition of the restriction with the transfer $\tau : H_H^3(S^3, H_2(T^*; \mathbb{Z})) \rightarrow H_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z}))$ is just a multiplication by the index $[\mathbb{Q}_{4n} : H]$,

$$\begin{array}{ccccc} H_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z})) & \rightarrow & H_H^3(S^3, H_2(T^*; \mathbb{Z})) & \rightarrow & H_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z})) \\ [c_{\mathbb{Q}_{4n}}(h)] & \mapsto & [c_H(h)] & \mapsto & [\mathbb{Q}_{4n} : H] \cdot [c_{\mathbb{Q}_{4n}}(h)]. \end{array}$$

Particularly, let H be the trivial subgroup of \mathbb{Q}_{4n} . The sphere S^3 is 2-connected and H is a trivial group, so the map h can be extended to a H -map $S^3 \rightarrow M$. Thus, $[c_H(h)] = 0$ and consequently $[\mathbb{Q}_{4n} : H] \cdot [c_{\mathbb{Q}_{4n}}(h)] = 0$ in $H_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z}))$. ■

3.4 The \mathbb{Q}_{4n} cellular structures on S^3

In order to start efficient computations of the obstruction cocycle we need to describe the concrete \mathbb{Q}_{4n} CW-structures of the sphere S^3 . The proposition 7 allows us to be very picky in selecting the adequate cellular structures. We describe two \mathbb{Q}_{4n} cellular structures the natural one and the most economical one, and the cellular map joining them. The direct consequence of these discussions is the isomorphism

$$H_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z})) \cong H_2(T^*; \mathbb{Z})_{\mathbb{Q}_{4n}}.$$

The natural \mathbb{Q}_{4n} cellular-simplicial structure. This structure comes from the join decomposition $S^3 = S^1 * S^1$ of the 3-sphere. Let the sphere S^1 be represented by the simplicial complex of the regular $2n$ -gon P_{2n} . Then the sphere S^3 , as the simplicial complex, is the join $P_{2n}^{(1)} * P_{2n}^{(2)}$ of two copies of P_{2n} . Let the vertex of $P_{2n}^{(1)}$ and $P_{2n}^{(2)}$ be denoted by a_1, \dots, a_{2n} and b_1, \dots, b_{2n} , respectively. The action of the group \mathbb{Q}_{4n} on S^3 is defined on vertices by

$$\epsilon \cdot a_i = a_{i \bmod 2n+1}, \epsilon \cdot b_i = b_{i \bmod 2n+1}, j \cdot a_1 = b_1$$

and it extends equivariantly to upper skeletons. Then, for example

$$\begin{aligned} j \cdot a_i &= j \epsilon^{i-1} \cdot a_1 = \epsilon^{(2n-1)(i-1)} j \cdot a_1 = \epsilon^{(2n-1)(i-1)} \cdot b_1 = b_{(2n-i+1) \bmod 2n+1} \\ j \cdot [a_1, a_2; b_1, b_2] &= [b_1, b_{2n}; a_{n+1}, a_n]. \end{aligned}$$

The associated chain complex $\mathfrak{C} = \{C_i\}$ has the form

$$0 \longrightarrow \mathbb{Z}^{4n^2} \longrightarrow \mathbb{Z}^{8n^2} \longrightarrow \mathbb{Z}^{4n^2+4n} \longrightarrow \mathbb{Z}^{4n} \longrightarrow 0.$$

This \mathbb{Q}_{4n} cellular structure makes the beautiful rectangular middle section of the join representation $S^3 = S^1 * S^1 = [0, 1] * [0, 1] / \approx$, (with some identifications). The action is indicated in the figure 1.

The economic \mathbb{Q}_{4n} cellular structure. The second one comes from the minimal resolution of \mathbb{Z} by free \mathbb{Q}_{4n} -modules described in [6], p. 253. The associated cellular complex has

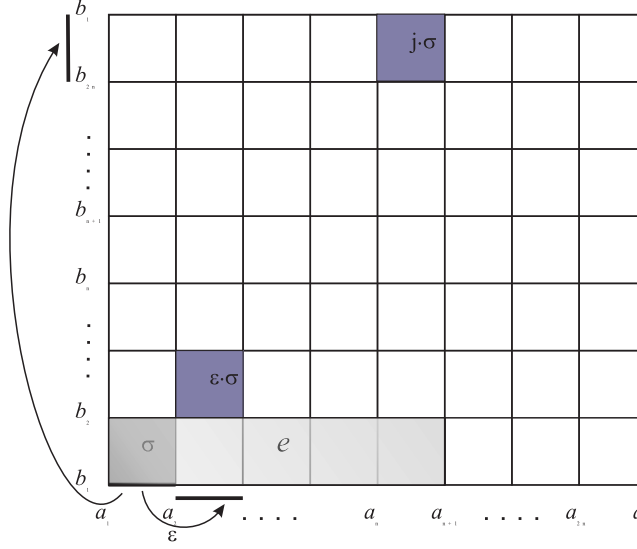


Figure 1: The model for two \mathbb{Q}_{4n} cellular structures of S^3 .

one \mathbb{Q}_{4n} 0-cell a , two \mathbb{Q}_{4n} 1-cells b and b' , two \mathbb{Q}_{4n} 2-cells c and c' , and finally then one \mathbb{Q}_{4n} 3-cell e . The associated chain complex $\mathfrak{D} = \{D_i\}$ has the form

$$0 \rightarrow \mathbb{Z}[\mathbb{Q}_{4n}]e \xrightarrow{\partial} \mathbb{Z}[\mathbb{Q}_{4n}]c \oplus \mathbb{Z}[\mathbb{Q}_{4n}]c' \xrightarrow{\partial} \mathbb{Z}[\mathbb{Q}_{4n}]b \oplus \mathbb{Z}[\mathbb{Q}_{4n}]b' \xrightarrow{\partial} \mathbb{Z}[\mathbb{Q}_{4n}]a \rightarrow 0$$

where

$$\begin{aligned} \partial e &= (\epsilon - 1)c - (\epsilon j - 1)c' & \partial c &= (1 + \dots + \epsilon^{n-1})b - (j + 1)b' & \partial c' &= (\epsilon j + 1)b + (\epsilon - 1)b' \\ \partial b &= (\epsilon - 1)a & \partial b' &= (j - 1)a \end{aligned}.$$

Thus, it will be enough to look at the obstruction cocycle $c_{\mathbb{Q}_{4n}}(h)$ on the maximal cell e , and to prove that its image is or is not zero, when we pass to cohomology.

Computing equivariant cohomology. The explicit formulas for the chain complex $\{D_i\}$ is lurking us to apply the $\text{Hom}_{\mathbb{Q}_{4n}}(\cdot, H_2(T^*; \mathbb{Z}))$ functor. The result is the equivariant cochain complex

$$0 \leftarrow H_2(T^*; \mathbb{Z}) \xleftarrow{\Gamma} H_2(T^*; \mathbb{Z}) \oplus H_2(T^*; \mathbb{Z}) \leftarrow H_2(T^*; \mathbb{Z}) \oplus H_2(T^*; \mathbb{Z}) \leftarrow H_2(T^*; \mathbb{Z}) \leftarrow 0$$

where $\Gamma(p, q) = (\epsilon - 1)p - (\epsilon j - 1)q$ for $p, q \in H_2(T^*; \mathbb{Z})$. The definition of the equivariant cohomology and a standard calculation imply that

$$H_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z})) = H_2(T^*; \mathbb{Z}) / \text{Im} \Gamma \cong H_2(T^*; \mathbb{Z})_{\mathbb{Q}_{4n}}$$

where $H_2(T^*; \mathbb{Z})_{\mathbb{Q}_{4n}}$ denote the group of coinvariants of the \mathbb{Q}_{4n} -module $H_2(T^*; \mathbb{Z})$, [5].

The chain map. There exist a cellular map $f : \mathfrak{D} \rightarrow \mathfrak{C}$, which can be visualized on the figure 1 by identifying top dimensional cell e of the economic cell structure with the transparent-shaded fundamental domain.

Since we are interested in 3-cochains, i.e. elements of $C_{\mathbb{Q}_{4n}}^3(S^3, H_2(T^*; \mathbb{Z}))$, we only need to know the concrete expression for the cellular map on the top dimensional cell e ,

$$f(e) = [a_1, \epsilon a_1] * [b_1, \epsilon b_1] + [\epsilon a_1, \epsilon^2 a_1] * [b_1, \epsilon b_1] + \dots + [\epsilon^{n-1} a_1, \epsilon^n a_1] * [b_1, \epsilon b_1],$$

where the simplexes on the right hand side are appropriately oriented.

3.5 Point classes

It is of utmost importance for computation of the obstruction cocycle $c(h)_{\mathbb{Q}_{4n}}$ to pinpoint some elements from the coefficient module $H_2(T^*; \mathbb{Z})$. The similar discussion can be found in [4].

Let $\{W_1, W_2, \dots, W_k\}$ be the family on maximal elements of the arrangement \mathcal{A} of linear (closed half-)subspaces in an $(n + m)$ -dimensional, Euclidean space E . Let also maximal elements have the constant dimension $\dim W_i = n$. Let $\hat{D}(\mathcal{A}) = \cup \mathcal{A} \cup \{+\infty\} \subset E \cup \{+\infty\} \cong S^{n+m}$ be the compactified union and $M(\mathcal{A}) = E \setminus \cup \mathcal{A}$ the complement of the arrangement. Let the point $x \in \text{int}(W_i \setminus \cup_{j \neq i} W_j)$ and let $D_\varepsilon(x) = x + D_\varepsilon$ be a disc around x , where D_ε is the ε -disc in the orthogonal complement W_i^\perp . For a sufficiently small ε the intersection $D_\varepsilon(x) \cap (\cup_{j \neq i} W_j)$ vanishes and we assume fix such an ε . We assume that $D_\varepsilon(x)$ is oriented by the orientation inherited from the ambient orientation and the orientation prescribed of W_i .

The point class $[x] \in H_m(E, M(\mathcal{A}); \mathbb{Z})$ of x is inclusion image of the fundamental class of the pair $(D_\varepsilon(x), \partial D_\varepsilon(x))$. By the Excision axiom described ε has no effect on the class $[x]$. In addition, by the Homotopy axiom, $[x]$ is uniquely determined by the connected component of $W_i \setminus \cup_{j \neq i} W_j$. The image $\|x\| := \partial[x]$ of the point class $[x]$ by the isomorphism $H_m(E, M(\mathcal{A}); \mathbb{Z}) \rightarrow H_{m-1}(M(\mathcal{A}); \mathbb{Z})$ is also called the *point class* of x and has all the properties of the original one.

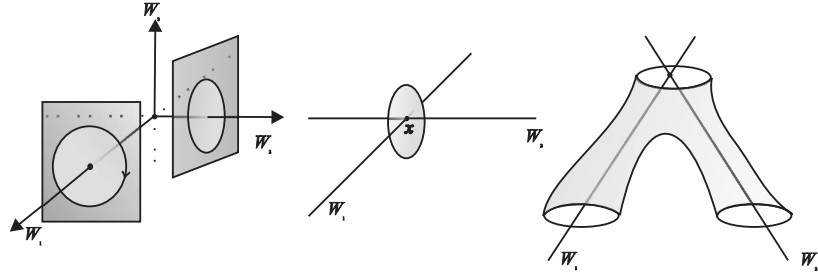


Figure 2: The point classes and the broken point classes

Proposition 9 *Let \mathcal{A} be the arrangement of linear subspaces as above. Let $x \in \text{int}(W_i \setminus \cup_{j \neq i} W_j) = W_i \setminus \cup_{j \neq i} W_j$.*

- (A) *The class $[x] \in H_m(E, M(\mathcal{A}); \mathbb{Z})$ does not vanish.*
- (B) *If we assume that $(\forall j) i \neq j \Rightarrow \text{codim}_{W_i}(W_i \cap W_j) > 1$, then the class $[x]$ does not depend on particular x .*
- (C) *If there is a subspace W_j such that $\text{codim}_{W_i}(W_i \cap W_j) = 1$ and x_1, x_2 belong to different connected components of $W_i \setminus \cup_{j \neq i} W_j$, then $[x_1] \neq [x_2]$.*

Proof. (A) By the Ziegler-Živaljević formula [19], has he wedge decomposition

$$\hat{D}(\mathcal{A}) \simeq \hat{W}_1 \vee \hat{W}_2 \vee \dots \vee \hat{W}_k \vee \dots$$

where the displayed factors correspond to maximal elements. Let $\Delta(\hat{W}_i) \in H^m(E, M(\mathcal{A}); \mathbb{Z})$ be the Poincaré-Alexander dual of the fundamental homology class $[\hat{W}_i] \in H_n(\hat{D}(\mathcal{A}); \mathbb{Z})$ associated with the sphere \hat{W}_i in \hat{E} . Then $\Delta(\hat{W}_i)([N]) \in \mathbb{Z}$ is the intersection number $[\hat{W}_i] \cap [N]$, whenever this number is correctly defined, for example if N is a manifold and the intersection $\hat{W}_i \cap N$ is transversal. From here we see that $\Delta(\hat{W}_i)([x]) = 1$ and so $[x]$ does not vanish.

(B) Since the assumption implies the connectness of the complement $W_i \setminus \cup_{j \neq i} W_j$, the statement follows by some homotopy.

(C) When $\text{codim}_{W_i}(W_i \cap W_j) = 1$ the subspaces W_i and W_j are decomposed by the hyperplane $W_i \cap W_j$ into unions of closed half-spaces, $W_i = W_i^1 \cup W_i^2$ and $W_j = W_j^1 \cup W_j^2$, respectively. Let us assume that $x_1 \in W_i^1$ and $x_2 \in W_i^2$. The half-space W_i^1 and W_j^1 glued along the common boundary and compactified determine a sphere $S \subset \hat{E}$. Since the Ziegler-Živaljević decomposition [19] involves a choice of generic points, the points can be chosen in such a way that the sphere S appears in the decomposition. Precisely, the sphere S is the factor $\widehat{W_i \cap W_j} * \Delta(P_{< W_i \cap W_j}) \cong S^{n-1} * S^0 \cong S^n$, where P denote the intersection poset of the arrangement \mathcal{A} . thus, the fundamental class $[S]$ is nontrivial in $H_n(\hat{D}(\mathcal{A}); \mathbb{Z})$. Like in the previous case, let $\Delta(S) \in H^m(E, M(\mathcal{A}); \mathbb{Z})$ be the Poincaré-Alexander dual to $[S]$. Then

$$\Delta(S)([x_1]) = \pm 1 \text{ and } \Delta(S)([x_2]) = 0$$

and consequently $[x_1] \neq [x_2]$. ■

The broken point class. In order to simplify the exposition let \mathcal{A} be a very special arrangement consisting of two vector spaces W_1 and W_2 of the same dimension m inside the vector space $E = \mathbb{R}^{n+m}$. Let $U = W_1 \cap W_2$ be the intersection and $k = \dim(W_1 \cap W_2)$. Let us also fix V , a vector space of dimension n such that $E = W_1 \oplus V = W_2 \oplus V$, where \oplus denotes the direct sum of vector spaces. For $x \in W_1 \cap W_2$ let $D_\varepsilon(x)$ be a disc $x + D_\varepsilon$, where D_ε is the ε -disc in V . Then the *broken point class* of x , denoted by $[x] \in H_n(E, E \setminus W_1 \cup W_2; \mathbb{Z})$, is the fundamental class of the pair $(D_\varepsilon(x), \partial D_\varepsilon(x))$. In addition the image $\|x\| := \partial[x]$ of the broken point class $[x]$ by the isomorphism $H_n(E, E \setminus W_1 \cup W_2; \mathbb{Z}) \rightarrow H_{n-1}(E \setminus W_1 \cup W_2; \mathbb{Z})$ is also called the broken point class of x and has all the properties of the original one.

An illustration for the case $m = 1$ and $n = 2$ can be seen in the figure 2. The same definition stands even if W_1 and W_2 are closed half spaces intersecting over linear space U . The broken point classes can be naturally expressed as linear combinations of the ordinary point classes. The third picture in the figure 2 illustrates this situation.

Let us now suppose that $k = \dim(W_1 \cap W_2) = \dim(W_1) - 1 = m - 1$. Then there are two different broken point classes associated to every point $x \in W_1 \cap W_2$, i.e. the broken point class $[x]$ depends on the choice of the vector space V . Let us omit the formal proof of this fact and illustrate this situation by the figure 3. The formal proof would go along the lines of the proposition 9. The main property that differs these two classes, brusquely explained, is that if we substitute each sphere with the disc and move these discs a little bit:

- (A) the first disc will intersect simultaneously W_1^+ and W_2^+ , or W_1^- and W_2^- ; and
- (B) the second disc will intersect simultaneously W_1^+ and W_2^- , or W_1^- and W_2^+ .

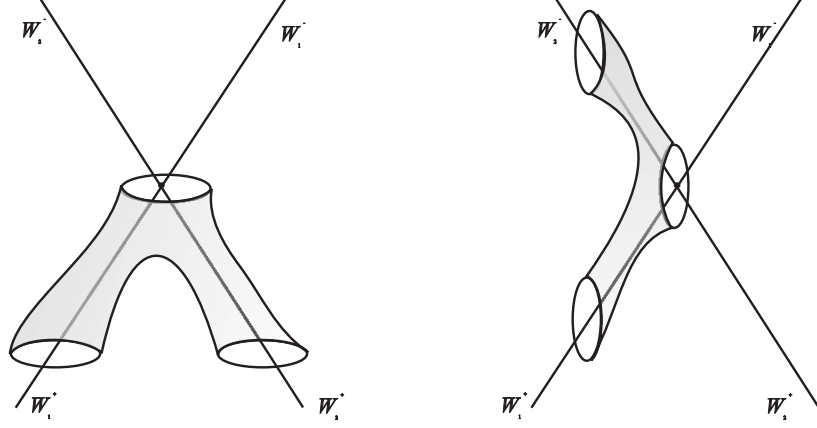


Figure 3: Two possible point classes in the case $\dim(W1 \cap W2) = m - 1$

3.6 Calculating $H_2(T^*; \mathbb{Z})$ and $H_2(T^*; \mathbb{Z})_{\mathbb{Q}_{4n}}$

The homology. In order to analyze the second homology of the complement T^* we use the following equality

$$W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha) = S^{n-1} \setminus \cup \hat{\mathcal{A}}(\mathcal{J}, \alpha)$$

where $\hat{\mathcal{A}}(\mathcal{J}, \alpha)$ denotes the one point compactification of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$. This allow us to use the Poincaré-Alexander duality and to work with the arrangement $\hat{\mathcal{A}}(\mathcal{J}, \alpha)$ instead of its complement. The Poincaré-Alexander duality and the Universal Coefficient isomorphisms give us the sequence of isomorphisms (assuming \mathbb{Z} coefficients)

$$\begin{aligned} H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha)) &= H_2(S^{n-1} \setminus \cup \hat{\mathcal{A}}(\mathcal{J}, \alpha)) \cong H^{(n-1)-2-1}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha)) \\ &\cong \text{Hom}(H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha)), \mathbb{Z}) \oplus \text{Ext}(H_{n-3}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha)), \mathbb{Z}). \end{aligned} \quad (1)$$

Since the maximal elements of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$ are $(n-4)$ -dimensional linear (half-) subspaces, respectively, the Ext factor in the above equality vanish. Precisely, $H_{n-3}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z}) = 0$. Therefore,

$$H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z}) \cong \text{Hom}(H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z}), \mathbb{Z}). \quad (2)$$

The Ziegler-Živaljević formula implies the following homology decomposition (assuming \mathbb{Z} coefficients)

$$H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z}) \cong \bigoplus_{d=0}^{n-4} \bigoplus_{V \in P(\alpha): \dim V=d} H_{n-4}(\Delta(P_{<V}) * \hat{V}) \quad (3)$$

where $P(\alpha)$ is the intersection poset of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$ and \hat{V} one-point compactification of the element V . Thus, the property of the hom functor imply

$$H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z}) \cong \bigoplus_{d=0}^{n-4} \text{Hom} \left(\bigoplus_{V \in P(\alpha): \dim V=d} H_{n-4}(\Delta(P_{<V}) * \hat{V}); \mathbb{Z} \right) \quad (4)$$

Poincaré dual of the (broken) point class. Let us illustrate the isomorphism (2) on the point classes for the general arrangement of linear (half-) spaces \mathcal{A} in \mathbb{R}^{n+m} . Let maximal elements $\{W_1, W_2, \dots, W_k\}$ have the constant dimension $\dim W_i = n$. Let $x \in W_1 \setminus \cup_{i \neq 1} W_i$ and $S = \partial D_\varepsilon(x)$, where $D_\varepsilon(x) = x + D_\varepsilon$ and D_ε is a small disk in W_1^\perp . Then the isomorphism $\vartheta : H_{m-1}(\mathbb{R}^{n+m} \setminus \cup \mathcal{A}; \mathbb{Z}) \rightarrow \text{Hom}(H_n(\cup \hat{\mathcal{A}}; \mathbb{Z}), \mathbb{Z})$ can be expressed for $t \in H_n(\cup \hat{\mathcal{A}}; \mathbb{Z})$ by

$$\vartheta(\|x\|) : H_n(\cup \hat{\mathcal{A}}; \mathbb{Z}) \rightarrow \mathbb{Z}, \vartheta(\|x\|)(t) = \text{link}(S, T) \quad (5)$$

if T is a submanifold in $\cup \hat{\mathcal{A}}$ representing the homology class t and linking number $\text{link}(S, T)$ is correctly defined.

Example 10 Let the arrangement \mathcal{A} be given by the figure 4,(A). Then the Hasse diagram of the arrangement \mathcal{A} is like the in the figure 4,(B).

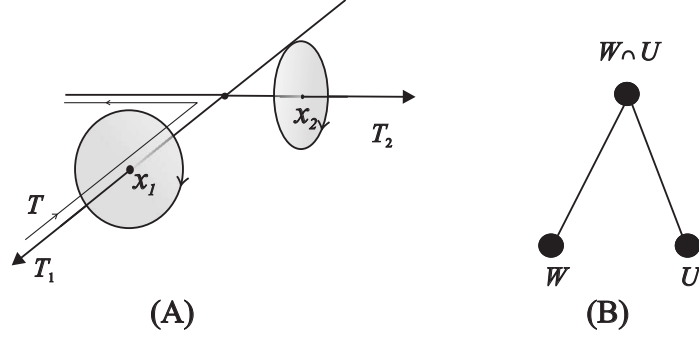


Figure 4: The arrangement \mathcal{A} with indicated generators in homology.

Then $H_1(\cup \hat{\mathcal{A}}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the generators / spheres T, T_1 and T_2 indicated in the figure. Then we can read of isomorphism $\vartheta : H_1(\mathbb{R}^3 \setminus \cup \mathcal{A}; \mathbb{Z}) \rightarrow \text{Hom}(H_1(\cup \hat{\mathcal{A}}; \mathbb{Z}), \mathbb{Z})$ for point classes $\|x_1\|$ and $\|x_2\|$ from the picture,

$$\begin{aligned} \vartheta(\|x_1\|)(t_1) &= 1, \vartheta(\|x_1\|)(t_2) = 0, \vartheta(\|x_1\|)(t) = -1, \\ \vartheta(\|x_2\|)(t_1) &= 0, \vartheta(\|x_2\|)(t_2) = 1, \vartheta(\|x_2\|)(t) = 0. \end{aligned}$$

The coinvariants. When time comes to compute coinvariants the best news would be that the Poincaré-Alexander duality map and the Universal coefficient isomorphism are equivariant maps. Then the isomorphisms (2) would be isomorphisms of \mathbb{Q}_{4n} -modules, and there would be no difference in what module we work. (Un)fortunately, the Poincaré-Alexander duality map is not a \mathbb{Q}_{4n} -map, but a \mathbb{Q}_{4n} -map up to a orientation character, while Universal coefficient isomorphism is an \mathbb{Q}_{4n} -map. If o is an orientation of the \mathbb{Q}_{4n} -sphere S^{n-1} or $S^{2(n-1)}$, then o determines Poincaré-Alexander duality map γ_o [10]. In particular, $g \cdot o = \det(g) \cdot o$, where $g \in \mathbb{Q}_{4n} \subseteq GL_n(\mathbb{R})$, or $g \in \mathbb{Q}_{4n} \subseteq GL_{2n}(\mathbb{R})$.

The natural \mathbb{Q}_{4n} -action on the union $\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha)$, inherited from the ambient \mathbb{Q}_{4n} -action, respects the dimensional decomposition (3) of the $(n-4)$ -homology. When these homologies

are free, then the decomposition (4) becomes true even without Hom. Let us assume that $(n-4)$ -homology of the arrangement $\widehat{\mathcal{A}}(\mathcal{J}, \alpha)$ is free. In some calculation we will see that we actually do not need the whole group to be free, but just some factors in the decomposition (4). Therefore, if we would rather work with \mathbb{Q}_{4n} -module $H_{n-4}(\cup \widehat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ instead of \mathbb{Q}_{4n} -module $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})$ we have to modify the \mathbb{Q}_{4n} -action. Specifically, let $l \in H_{n-4}(\cup \widehat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ and $g \in \mathbb{Q}_{4n}$, then

$$g * l = \det(g) g \cdot l$$

where $*$ is the new modified action, and \cdot the old one. Let \sim denote the congruence relation on $H_{n-4}(\cup \widehat{\mathcal{A}}(\mathcal{J}, \alpha), \mathbb{Z})$ which class of zero is a subgroup generated by the elements of the form $g * x - x$, $g \in \mathbb{Q}_{4n}$, $x \in H_{n-4}(\widehat{\mathcal{A}}(\mathcal{J}, \alpha), \mathbb{Z})$. Then there is an isomorphism

$$H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}} \cong H_{n-4}(\cup \widehat{\mathcal{A}}(\mathcal{J}, \alpha), \mathbb{Z}) / \sim \quad (6)$$

The torsion of the coinvariants. The proposition 8 directs us to search for the obstruction cohomology class in the torsion part of the coinvariants. Therefore, if the coinvariants are free, then the obstruction is zero and the map exists. Thus everything we have done is in vain. It is of utmost importance to get a feeling, when some \mathbb{G} arrangement can produce nontrivial torsion group in appropriate coinvariant group.

Let us discuss a few simple examples, which met all the above assumptions. These examples come from the fan partition problems, and we work with cyclic groups in order to simplify computations.

Example 11 Let \mathcal{A} be the minimal \mathbb{Z}_8 arrangement in \mathbb{R}^8 containing subspace

$$L = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 0\} \subset W_8.$$

The group $\mathbb{Z}_8 = \langle \varepsilon \rangle$ acts by a cyclic permutation, i.e. $\varepsilon \cdot (x_1, \dots, x_8) = (x_8, x_1, \dots, x_7)$. Then $\det(\varepsilon) = (-1)^{8+1}$.

It is not hard to see that $\mathcal{A} = \{L, \varepsilon L, L \cap \varepsilon L\}$, $L \cap \varepsilon L = \{0\}$, and consequently the by Z-Z decomposition (3),

$$H_4(\cup \widehat{\mathcal{A}}; \mathbb{Z}) \cong \bigoplus_{d=0V \in P: \dim V=d}^4 \bigoplus H_4(\Delta(P_{<V}) * \widehat{V}; \mathbb{Z}) \cong 0 \oplus 0 \oplus 0 \oplus (\mathbb{Z} \oplus \mathbb{Z})$$

where P is the intersection poset of the arrangement \mathcal{A} . Since, homology $H_4(\cup \widehat{\mathcal{A}}; \mathbb{Z})$ has no torsion we have that $H_2(W_8 \setminus \cup \mathcal{A}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let l and $\varepsilon \cdot l$ be the generators of $H_4(\cup \widehat{\mathcal{A}}; \mathbb{Z})$ corresponding to spheres \widehat{L} and $\widehat{\varepsilon L}$. The equality $L = \varepsilon^2 L$ in \mathbb{R}^8 imply the equality $l = \epsilon(\varepsilon^2 \cdot l)$ in $H_4(\cup \widehat{\mathcal{A}}; \mathbb{Z})$, where $\epsilon \in \{1, -1\}$. The sign ϵ depends on the nature of ε^2 . If ε^2 changes the orientation of L , then $\epsilon = -1$, otherwise $\epsilon = 1$. To calculate the sign ϵ we use a ε^2 -invariant decomposition $\mathbb{R}^8 = L \oplus L^\perp$, where L^\perp denote the orthogonal complement of L . Then,

$$\det_{\mathbb{R}^8} \varepsilon^2 = \det_L \varepsilon^2 \cdot \det_{L^\perp} \varepsilon^2 \Rightarrow \det_L \varepsilon^2 = \det_{\mathbb{R}^8} \varepsilon^2 \det_{L^\perp} \varepsilon^2 = (-1)^{2 \cdot (8+1)} \det_{L^\perp} \varepsilon^2.$$

Let e_1, \dots, e_8 be the standard base of \mathbb{R}^8 . Then one base for L^\perp is

$$f_1 = e_1 + e_2, f_2 = e_3 + e_4, f_3 = e_5 + e_6, f_4 = e_1 + e_2 + \dots + e_7 + e_8,$$

and ε^2 acts on it by

$$\varepsilon^2 \cdot f_1 = f_2, \varepsilon^2 \cdot f_2 = f_3, \varepsilon^2 \cdot f_3 = f_4 - f_1 - f_2 - f_3, \varepsilon^2 \cdot f_4 = f_4.$$

Therefore,

$$\det_{L^\perp} \varepsilon^2 = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -1 \Rightarrow \epsilon = \det_L \varepsilon^2 = -\det_{\mathbb{R}^8} \varepsilon^2 = -1 \Rightarrow l = -(\varepsilon^2 \cdot l)$$

Now, we calculate coinvariants $H_2(W_n \setminus \cup \mathcal{A}; \mathbb{Z})_{\mathbb{Z}_8}$ by using the modified action $g * l = \det(g) g \cdot l$, $g \in \mathbb{Z}_8$ on \mathbb{Z}_8 -module $H_4(\cup \hat{\mathcal{A}}; \mathbb{Z})$. Let us observe that

$$l \sim \varepsilon * l = \det(\varepsilon) \varepsilon \cdot l = -\varepsilon \cdot l \text{ and } l \sim \varepsilon^2 * l = \det(\varepsilon^2) \varepsilon^2 \cdot l = -\det(\varepsilon^2) \det(\varepsilon^2) l = -l.$$

Thus, from this relations we can conclude that $H_2(W_n \setminus \cup \mathcal{A}; \mathbb{Z})_{\mathbb{Z}_8} \cong \mathbb{Z}_2$.

Example 12 Let \mathcal{B} be the minimal \mathbb{Z}_4 arrangement in $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$ containing subspace

$$L = \{\mathbf{x} \in \mathbb{R}^8 \mid x_1 = x_5 = x_1 + \dots + x_4 = x_5 + \dots + x_8 = x_3 + x_7 = 0\} \subset W_4 \oplus W_4.$$

Let the action of $\mathbb{Z}_4 = \langle \varepsilon \rangle$ be defined by $\varepsilon \cdot (x_1, \dots, x_8) = (x_4, x_1, x_2, x_3; x_8, x_5, x_6, x_7)$. Then $\det(\varepsilon) = (-1)^{4+1}(-1)^{4+1} = 1$. The arrangement \mathcal{A} has 4 maximal elements $L, \varepsilon L, \varepsilon^2 L, \varepsilon^3 L$ and the Hasse diagram of the intersection poset P is like in the figure 5.

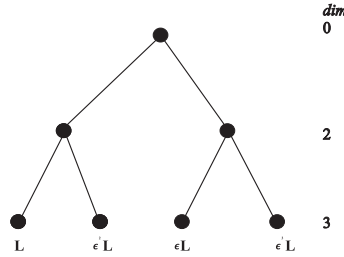


Figure 5: The Hasse diagram of \mathcal{B} .

$$H_3(\cup \hat{\mathcal{B}}; \mathbb{Z}) \cong \bigoplus_{d=0}^3 \bigoplus_{V \in Q(\alpha): \dim V=d} H_{2n-5}(\Delta(P_{<V}) * \hat{V}) \cong 0 \oplus (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}).$$

Like in the previous example the homology $H_3(\cup \hat{\mathcal{B}}; \mathbb{Z})$ has no torsion and so $H_2(W_4 \oplus W_4 \setminus \cup \mathcal{B}; \mathbb{Z}) \cong H_3(\cup \hat{\mathcal{B}}; \mathbb{Z})$.

Let $l, \varepsilon \cdot l, \varepsilon^2 \cdot l, \varepsilon^3 \cdot l$ and $k, \varepsilon \cdot k$ be the generators of $H_3(\cup \widehat{\mathcal{B}}; \mathbb{Z})$ corresponding to spheres $\widehat{L}, \widehat{\varepsilon L}, \widehat{\varepsilon^2 L}, \widehat{\varepsilon^3 L}$ and $\widehat{L \cap \varepsilon^2 L * S^0}, \widehat{\varepsilon L \cap \varepsilon^3 L * S^0}$ respectively. The equality $L \cap \varepsilon^2 L = \varepsilon^2(L \cap \varepsilon^2 L)$ in \mathbb{R}^8 imply the equality $k = \epsilon(\varepsilon^2 k)$, where $\epsilon \in \{1, -1\}$. The sign ϵ depends on the nature of ε^2 . If ε^2 changes the orientation of the sphere $\widehat{L \cap \varepsilon^2 L * S^0}$, then $\epsilon = -1$, otherwise $\epsilon = 1$. Again, we use the same decomposition $\mathbb{R}^8 = (L \cap \varepsilon^2 L) \oplus (L \cap \varepsilon^2 L)^\perp$ and calculate

$$\det_{\mathbb{R}^8} \varepsilon^2 = \det_{(L \cap \varepsilon^2 L)} \varepsilon^2 \cdot \det_{(L \cap \varepsilon^2 L)^\perp} \varepsilon^2 \Rightarrow \det_{(L \cap \varepsilon^2 L)} \varepsilon^2 = \det_{\mathbb{R}^8} \varepsilon^2 \det_{(L \cap \varepsilon^2 L)^\perp} \varepsilon^2 = \det_{(L \cap \varepsilon^2 L)^\perp} \varepsilon^2.$$

If e_1, \dots, e_8 is the standard base of \mathbb{R}^8 , then one base for $(L \cap \varepsilon^2 L)^\perp$ is

$$\begin{aligned} f_1 &= e_1, f_2 = e_1 + e_2 + e_3 + e_4, f_3 = e_5, \\ f_4 &= e_5 + e_6 + e_7 + e_8, f_5 = e_3, f_6 = e_7, \end{aligned}$$

and ε^2 acts on it by

$$\varepsilon^2 \cdot f_1 = f_5, \varepsilon^2 \cdot f_2 = f_2, \varepsilon^2 \cdot f_3 = f_6, \varepsilon^2 \cdot f_4 = f_4, \varepsilon^2 \cdot f_5 = f_1, \varepsilon^2 \cdot f_6 = f_3.$$

Since the sign of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 4 & 1 & 3 \end{pmatrix}$ is one then $\det_{(L \cap \varepsilon^2 L)^\perp} \varepsilon^2 = 1$.

In order to verify that the element ε^2 does not change the orientation of the sphere $\widehat{L \cap \varepsilon^2 L * S^0}$, we analyze the action of ε^2 on the homology $H_3(\widehat{L \cap \varepsilon^2 L * S^0}; \mathbb{Z})$. Since $\widehat{L \cap \varepsilon^2 L}$ is a 2-sphere and, we saw, ε^2 acts trivially on it, the isomorphism

$$H_3(\widehat{L \cap \varepsilon^2 L * S^0}; \mathbb{Z}) \cong \tilde{H}_0(S^0; \mathbb{Z}) \otimes H_2(\widehat{L \cap \varepsilon^2 L}; \mathbb{Z})$$

instructs us that it remains to look at the action of the ε^2 on $\tilde{H}_0(S^0; \mathbb{Z})$. Keeping in mind that S^0 is the order complex of the lower cone of the element $L \cap \varepsilon^2 L$, it is obvious that ε^2 acts on $H_0(S^0; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ by permuting generators of the two copies of \mathbb{Z} and remembering the orientation, i. e.

$$(x, y) \mapsto \det \varepsilon^2(y, x)$$

Therefore, the definition of the augmentation implies that ε^2 acts by multiplication on the reduced homology $\tilde{H}_0(S^0; \mathbb{Z}) \cong \mathbb{Z}$. Thus, $k = (\det \varepsilon^2) \varepsilon^2 \cdot k$.

Again we compute the coinvariants $H_2(W_4 \oplus W_4 \setminus \cup \mathcal{B}; \mathbb{Z})_{\mathbb{Z}_4}$ using the modified action. The following relations

$$l \sim \varepsilon^i * l = \det(\varepsilon^i) \varepsilon^i \cdot l = \varepsilon^i \cdot l, k \sim \varepsilon * k = \det(\varepsilon) \varepsilon \cdot k = \varepsilon k, k \sim \varepsilon^2 * k = (\det \varepsilon^2) \varepsilon^2 \cdot k = k$$

imply that actually nothing "torsion-like" happens and that $H_2(W_4 \oplus W_4 \setminus \cup \mathcal{B}; \mathbb{Z})_{\mathbb{Z}_4} \cong \mathbb{Z} \oplus \mathbb{Z}$.

3.7 How to compute the obstruction cocycle

Finally, we are ready to give an algorithm for computing the cohomology class of the obstruction cocycle of the map in general position. Let us assume that additional hyper arrangement \mathcal{J} is already chosen.

Step 1: Let S^3 be \mathbb{Q}_{4n} the simplicial complex earlier defined. Define \mathbb{Q}_{4n} -map $h : S^3 \rightarrow W_n$ by defining them on 0-skeleton. It is enough to define an image of a single vertex, because everything else is defined by the equivariant request.

Step 2: Find all simplexes $\sigma_{ij} = [a_i, a_{i+1}; b_j, b_{j+1}]$ such that

$$h([a_i, a_{i+1}; b_j, b_{j+1}]) \cap (L(\alpha) \cap \mathcal{J}) = \{pt.\}.$$

The intersection can't have more than one point, because then at least the whole interval will be in the intersection. This would mean that the map h is not in the general position.

Step 3: With the help of the cellular map between two introduced \mathbb{Q}_{4n} cellular structures on S^3 , and previous step, count the following sets

$$h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha)) \text{ and } h^{-1}(h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha))) \subset e.$$

Step 4: Let us assume that every element $y \in h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha))$ is contained in just one and only one maximal element L_y of the arrangement \mathcal{A} . Then

$$c_{\mathbb{Q}_{4n}}(h)(e) = \sum_{x \in h^{-1}(h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha)))} I(e, L_{h(x)}) \|h(x)\| \quad (7)$$

where $I(h(\theta), L_{h(x)})$ is the intersection number of the oriented cell e and appropriate oriented element L_y ($L_y \cap h(\theta) = \{y\}$) of the arrangement $\mathcal{A}(J, \alpha)$.

If some $y \in h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha))$ belongs to codimension one intersection of maximal elements $L_y^{(1)}, \dots, L_y^{(k)}$ the formulas is unchanged except the class $\|h(x)\|$ is a broken point class. The real trouble with this situation is that we need an extra effort to identify this broken point class. As we mentioned in section 3.5 the broken point class depends on the embedding of the simplex (or its linear span) which intersects arrangement in $y \in h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha))$.

Step 5: Compute $H_{n-4}(\cup \hat{\mathcal{A}}(J, \alpha); \mathbb{Z})$ using decomposition (3). If there are no torsion we actually calculated $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})$.

Step 6: Compute the coinvariants $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}}$ by working in \mathbb{Q}_{4n} -module $H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ using modified action - isomorphism (6).

Step 7: Express the obstruction element $c_{\mathbb{Q}_{4n}}(h)(e) \in H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})$ as the element of the group $\text{Hom}(H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z}), \mathbb{Z})$ via the isomorphism (2).

Step 8: Identify the class of the obstruction cocycle $c_{\mathbb{Q}_{4n}}(h)(e)$ in the group

$$H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha), \mathbb{Z}) / \sim.$$

If it is not zero we proved that the \mathbb{Q}_{4n} -map in question can not exist. Thus, propositions 4 and 5 imply that the appropriate fan partition exists. If it is zero, then the whole effort was in vane.

4 Computations and proof of theorem 2

The proof of the theorem 2 has two stages

- The proof for the wide class of special cases; in particular we prove theorem for all $\alpha = (\frac{a}{n}, \frac{a+b}{n}, \frac{b}{n}) \in \frac{1}{n} \mathbb{N}^3 \subseteq \mathbb{Q}^3$ such that $2a + 2b = n$, $a, b \geq 1$. This proof goes along the lines described in section 3.7.
- The limit argument extend the result from the class of special cases to the whole class of triples $\alpha = (a, a + b, b) \in \mathbb{R}_{>0}^3$, $2a + 2b = 1$.

4.1 Finding hyper arrangement \mathcal{J}

Before we try to prove the main theorem in steps described in the section 3.7, let us make the fundamental step by defining the additional hyper arrangement \mathcal{J} in \mathbb{R}^n . Let us define hyperplanes H_1, H_2, K and half-spaces K^+, K^- , by

$$\begin{aligned} H_1 &= \{\mathbf{x} \in \mathbb{R}^n \mid (a+b)(x_a - x_{2a+b} + x_1 - x_{a+b+1}) + x_{a+1} - x_{2a+b+1} + x_n - x_{a+b} = 0\}, \\ H_2 &= \{\mathbf{x} \in \mathbb{R}^n \mid x_{a+1} + \dots + x_{a+b} = 0\}, K = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \dots + x_{a+b} = 0\} \\ K^+ &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \dots + x_{a+b} \geq 0\} \text{ and } K^- = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 + \dots + x_{a+b} \leq 0\}. \end{aligned}$$

The hyper arrangement \mathcal{J} we would like to consider is

$$\mathcal{J} = \{H_1 \cap K_1^+, H_2\}.$$

Let us consider some properties of this arrangement which will produce crucial arguments in the proof of the main theorem. The first property is that $\varepsilon^{a+b}H_1 = H_1$ and ε^{a+b} does change the orientation of the subspace H_1^\perp . Indeed, let $e = (a+b)(e_a - e_{2a+b} + e_1 - e_{a+b+1}) + e_{a+1} - e_{2a+b+1} + e_n - e_{a+b}$ be a base vector of H_1^\perp , then $\varepsilon^{a+b} \cdot e = (a+b)(e_{2a+b} - e_a + e_{a+b+1} - e_1) + e_{2a+b+1} - e_{a+1} + e_{a+b} - e_n$. The other property is that $\varepsilon^{a+b}(K^+ \cap W_n) = K^- \cap W_n$. Thus the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$ is the minimal \mathbb{Q}_{4n} -arrangement containing half-subspace $L_1^* = L(\alpha) \cap H_1 \cap K^+$ and subspace $L_2^* = L(\alpha) \cap H_2$ defined by

$$\begin{aligned} L_1^* &: \begin{cases} x_1 + \dots + x_a = x_{a+1} + \dots + x_{2a+b} = x_{2a+b+1} + \dots + x_n = 0, & x_1 + \dots + x_{a+b} \geq 0, \\ (a+b)(x_a - x_{2a+b} + x_1 - x_{a+b+1}) + x_{a+1} - x_{2a+b+1} + x_n - x_{a+b} = 0. \end{cases} \\ L_2^* &: x_1 + x_2 + \dots + x_a = x_{a+1} + \dots + x_{a+b} = x_{a+b+1} + \dots + x_{2a+b} = x_{2a+b+1} + \dots + x_n = 0. \end{aligned}$$

The introduced setting provides us with the following facts:

(i) The intersection

$$I = L_1^* \cap \varepsilon^{a+b}L_1^* \cap \varepsilon^a j L_1^* \cap \varepsilon^{2a+b} j L_1^* \cap L_2^*$$

is a linear subspace of codimension one in both $L_1^* = L(\alpha) \cap H_1 \cap K^+$ and $L_2^* = L(\alpha) \cap H_2$;

(ii) The following set of equalities stands

$$\varepsilon^{a+b}(L_1^* \cap \varepsilon^{a+b}L_1^*) = L_1^* \cap \varepsilon^{a+b}L_1^*, \quad \varepsilon^{a+b}(\varepsilon^a j L_1^* \cap \varepsilon^{2a+b} j L_1^*) = \varepsilon^a j L_1^* \cap \varepsilon^{2a+b} j L_1^*, \quad \varepsilon^{a+b}L_2^* = L_2^*.$$

(iii) The element ε^{a+b} changes the orientation on I^\perp .

These properties will be the key arguments that the appropriate group of coinvariants $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha), \mathbb{Z})_{\mathbb{Q}_{4n}}$ have the nontrivial torsion part. The part of our arrangement $\cup \mathcal{A}(\mathcal{J}, \alpha)$ can be pictured like in the figure 6.

4.2 Step 1

Let us recall that we substituted the sphere S^3 with the simplicial complex $P_{2n}^{(1)} * P_{2n}^{(2)}$, where P_{2n} is the regular $2n$ -gon. Earlier we have denoted vertices of two copies of P_{2n} by a_1, \dots, a_{2n} and b_1, \dots, b_{2n} , respectively. To simplify notation in calculation ahead let $t = a_1$. Let e_1, \dots, e_n be the standard base of \mathbb{R}^n and $u_i = e_i - \frac{1}{n} \sum_{j=1}^n e_j$, $i \in \{1, \dots, n\}$.

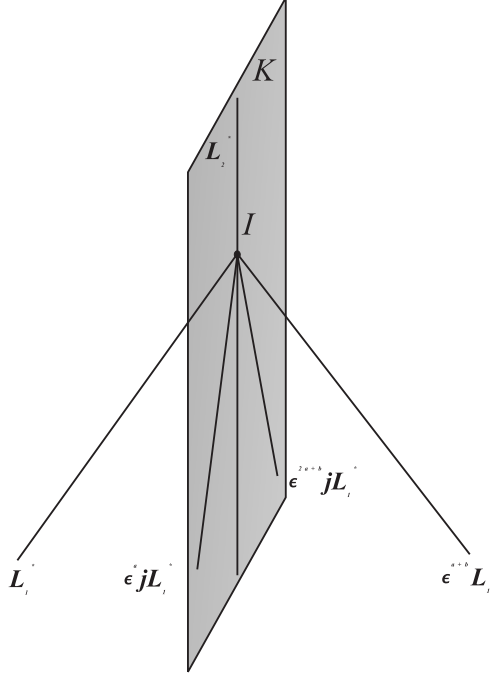


Figure 6: The part of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$

We define the \mathbb{Q}_{4n} -map $h : S^3 \rightarrow W_n \subset \mathbb{R}^n$ which is in the general position by defining it on the vertex t by $h(t) = u_1$. Then the request that h is \mathbb{Q}_{4n} -map imply

$$\begin{aligned} h(\varepsilon^i t) &= \varepsilon^i h(t) = \varepsilon^{i \bmod n} h(t) = u_{i \bmod n+1} \\ h(jt) &= jh(t) = u_n \\ h(\varepsilon^i jt) &= \varepsilon^i jh(t) = \varepsilon^{i \bmod n} jh(t) = u_{i \bmod n}. \end{aligned}$$

In the future we will omit the "mod n " part in the indexes on the right hand side, i.e. all the indexes in W_n are calculated mod n .

4.3 Step 2

It is not hard to see from the definition that the image of the map h is $h(S^3) = \bigcup_{i,j=1}^n [u_i, u_{i+1}] * [u_j, u_{j+1}] \subset W_n$, remembering that $n+1$ is actually 1. Let us list all the simplexes of the form $[u_i, u_{i+1}] * [u_j, u_{j+1}] \equiv [u_i, u_{i+1}; u_j, u_{j+1}]$ which intersect the subspace

$$L(\alpha) = \{\mathbf{x} \in W_n \mid x_1 + \dots + x_a = 0, x_{a+1} + \dots + x_{2a+b} = 0, x_{2a+b+1} + \dots + x_n = 0\}.$$

With a little linear algebra and combinatorics we see that the only simplexes from the image $h(S^3)$ that intersects $L(\alpha)$ are:

Simplex	Intersection with $L(\alpha)$	For $r \in$
1. $[u_a, u_{a+1}; u_{2a+b}, u_{2a+b+1}]$	$\{\frac{a}{n}u_a + \frac{b}{n}u_{a+1} + \frac{\gamma}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1}\}$	
2. $[u_a, u_{a+1}; u_r, u_{r+1}]$	$\{\frac{a}{n}u_a + \frac{a+b}{n}u_{a+1} + \frac{\gamma}{n}u_r + \frac{\delta}{n}u_{r+1}\}$	$[2a+b+1, n-1]$
3. $[u_r, u_{r+1}; u_{2a+b}, u_{2a+b+1}]$	$\{\frac{\alpha}{n}u_r + \frac{\beta}{n}u_{r+1} + \frac{a+b}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1}\}$	$[1, a-1]$
4. $[u_a, u_{a+1}; u_n, u_1]$	$\{\frac{\alpha}{n}u_a + \frac{a+b}{n}u_{a+1} + \frac{b}{n}u_n + \frac{\delta}{n}u_1\}$	
5. $[u_{2a+b}, u_{2a+b+1}; u_n, u_1]$	$\{\frac{a+b}{n}u_{2a+b} + \frac{\beta}{n}u_{2a+b+1} + \frac{\gamma}{n}u_n + \frac{a}{n}u_1\}$	
6. $[u_r, u_{r+1}; u_n, u_1]$	$\{\frac{\alpha}{n}u_r + \frac{\beta}{n}u_{r+1} + \frac{b}{n}u_n + \frac{a}{n}u_1\}$	$[a+1, 2a+b-1]$

Whit the assumption that $a, b \geq 1$ we analyze two cases.

(1) If we introduce the hyperplane H_1 , then the only simplex that intersects $L(\alpha) \cap H_1$ are

$$\rho_1 = [u_a, u_{a+1}; u_{2a+b}, u_{2a+b+1}], \rho_2 = [u_{a+b}, u_{a+b+1}; u_n, u_1], \rho_3 = [u_{a+b+1}, u_{a+b+2}; u_n, u_1].$$

Precisely,

$$\begin{aligned} (L(\alpha) \cap H_1) \cap \rho_1 &= \left\{ \frac{a}{n}u_a + \frac{b}{n}u_{a+1} + \frac{a}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1} \right\} \\ (L(\alpha) \cap H_1) \cap \rho_2 &= \left\{ \frac{b}{n}u_{a+b} + \frac{a}{n}u_{a+b+1} + \frac{b}{n}u_n + \frac{a}{n}u_1 \right\} \\ (L(\alpha) \cap H_1) \cap \rho_3 &= \left\{ \frac{a+\frac{b}{a+b}}{n}u_{a+b+1} + \frac{b-\frac{b}{a+b}}{n}u_{a+b+2} + \frac{b}{n}u_n + \frac{a}{n}u_1 \right\}. \end{aligned}$$

But when we add the inequality condition $x_1 + \dots + x_{a+b} \geq 0$, the intersection of $L_1^* = L(\alpha) \cap H_1 \cap K^+$ with the simplex ρ_3 vanishes. Thus there are only two simplexes

$$[u_a, u_{a+1}; u_{2a+b}, u_{2a+b+1}] \text{ and } [u_{a+b}, u_{a+b+1}; u_n, u_1]$$

which intersects L_1^* , each of them in the single interior point,

$$v = \frac{a}{n}u_a + \frac{b}{n}u_{a+1} + \frac{a}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1} \in L_1^* \text{ and } w = \frac{b}{n}u_{a+b} + \frac{a}{n}u_{a+b+1} + \frac{b}{n}u_n + \frac{a}{n}u_1 \in L_1^*.$$

Thus, $h(S^3) \cap L_1^* = \{v, w\}$.

(2) If we introduce the hyperplane H_2 instead the hyperplane H_1 , there are only two simplexes from $h(S^3)$ that intersects $L_2^* = L(\alpha) \cap H_2$. Those are

$$\rho_1 = [u_a, u_{a+1}; u_{2a+b}, u_{2a+b+1}] \text{ and } \rho_2 = [u_{a+b}, u_{a+b+1}; u_n, u_1]$$

and by another miracle they intersect L_2^* in the same points as in the previous case, $h(S^3) \cap L_2^* = \{v, w\}$.

Conclusion 13 *This means that in order to analyze the set $h^{-1}(h(e) \cap (\cup \mathcal{A}(\mathcal{J}, \alpha))) \subset e$ we only have to track down the pre images of v and w .*

4.4 Step 3

There are 16 simplexes in the sphere $P_{2n}^{(1)} * P_{2n}^{(2)}$ which belong to the h inverse image of the simplexes $[u_a, u_{a+1}; u_{2a+b}, u_{2a+b+1}]$ and $[u_{a+b}, u_{a+b+1}; u_n, u_1]$. These simplexes are

$$\begin{aligned} \sigma_1 &= [\epsilon^{a-1}t, \epsilon^at; \epsilon^{2a+b}jt, \epsilon^{2a+b+1}jt], & \sigma_2 &= [\epsilon^{n+a-1}t, \epsilon^{n+a}t; \epsilon^{2a+b}jt, \epsilon^{2a+b+1}jt], \\ \sigma_3 &= [\epsilon^{a-1}t, \epsilon^at; \epsilon^{n+2a+b}jt, \epsilon^{n+2a+b+1}jt], & \sigma_4 &= [\epsilon^{n+a-1}t, \epsilon^{n+a}t; \epsilon^{n+2a+b}jt, \epsilon^{n+2a+b+1}jt], \\ \sigma_5 &= [\epsilon^{2a+b-1}t, \epsilon^{2a+b}t; \epsilon^ajt, \epsilon^{a+1}jt], & \sigma_6 &= [\epsilon^{n+2a+b-1}t, \epsilon^{n+2a+b}t; \epsilon^ajt, \epsilon^{a+1}jt], \\ \sigma_7 &= [\epsilon^{2a+b-1}t, \epsilon^{2a+b}t; \epsilon^{n+a}jt, \epsilon^{n+a+1}jt], & \sigma_8 &= [\epsilon^{n+2a+b-1}t, \epsilon^{n+2a+b}t; \epsilon^{n+a}jt, \epsilon^{n+a+1}jt], \\ \theta_1 &= [\epsilon^{a+b-1}t, \epsilon^{a+b}t; jt, \epsilon jt], & \theta_2 &= [\epsilon^{n+a+b-1}t, \epsilon^{n+a+b}t; jt, \epsilon jt], \\ \theta_3 &= [\epsilon^{a+b-1}t, \epsilon^{a+b}t; \epsilon^n jt, \epsilon^{n+1}jt], & \theta_4 &= [\epsilon^{n+a+b-1}t, \epsilon^{n+a+b}t; \epsilon^n jt, \epsilon^{n+1}jt], \\ \theta_5 &= [\epsilon^{2n-1}t, t; \epsilon^{a+b}jt, \epsilon^{a+b+1}jt], & \theta_6 &= [\epsilon^{2n-1}t, t; \epsilon^{n+a+b}jt, \epsilon^{n+a+b+1}jt], \\ \theta_7 &= [\epsilon^{n-1}t, \epsilon^nt; \epsilon^{a+b}jt, \epsilon^{a+b+1}jt], & \theta_8 &= [\epsilon^{n-1}t, \epsilon^nt; \epsilon^{n+a+b}jt, \epsilon^{n+a+b+1}jt] \end{aligned}$$

By some miracle all these simplexes are in the orbit of the single simplex σ in the maximal cell e . Moreover, the pre-images of intersection points with L^*

$$v = \frac{a}{n}u_a + \frac{b}{n}u_{a+1} + \frac{a}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1} \text{ and } w = \frac{b}{n}u_{a+b} + \frac{a}{n}u_{a+b+1} + \frac{b}{n}u_n + \frac{a}{n}u_1$$

are all in the orbit of two points

$$v^* = \frac{a}{n}\epsilon^{a+b-1}t + \frac{b}{n}\epsilon^{a+b}t + \frac{a}{n}jt + \frac{b}{n}\epsilon jt \text{ and } w^* = \frac{b}{n}\epsilon^{a+b-1}t + \frac{a}{n}\epsilon^{a+b}t + \frac{b}{n}jt + \frac{a}{n}\epsilon jt$$

in the simplex σ . In particular,

$$\begin{aligned} \sigma &= [\epsilon^{a+b-1}t, \epsilon^{a+b}t; jt, \epsilon jt] = \epsilon j \epsilon^{-a+1} \sigma_1 = \epsilon^{-2a-b} \sigma_2 = \epsilon^{-n-2a-b} \sigma_3 = \epsilon j \epsilon^{-n-a+1} \sigma_4 \\ &= \epsilon^{-a} \sigma_5 = \epsilon j \epsilon^{-n-2a-b+1} \sigma_6 = \epsilon j \epsilon^{-2a-b+1} \sigma_7 = \epsilon^{-a-n} \sigma_8 \\ &= \theta_1 = \epsilon j \epsilon^{-n-a-b+1} \theta_2 = \epsilon j \epsilon^{-a-b+1} \theta_3 = \epsilon^{-n} \theta_4 \\ &= \epsilon j \epsilon \theta_5 = \epsilon^{-n-a-b} \theta_6 = \epsilon^{-a-b} \theta_7 = \epsilon j \epsilon^{-n+1} \theta_8 \end{aligned}$$

To illustrate the fact that points v^* and w^* from the cell e are mapped in the arrangement, we track the change of the barycentric coordinates of points v , w and v^* , w^* along the actions. For example,

(A) From $\sigma = [\epsilon jt, jt; \epsilon^{a+b-1}t, \epsilon^{a+b}t] = \epsilon j \epsilon^{-a+1} \sigma_1 = \epsilon^a j \sigma_1$, we have

$$\begin{aligned} h(\sigma_1) \cap L_1^* &= \left\{ \frac{a}{n}u_a + \frac{b}{n}u_{a+1} + \frac{a}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1} \right\} \\ h(\epsilon^a j \sigma_1) \cap \epsilon^a j L_1^* &= \left\{ \frac{a}{n}u_1 + \frac{b}{n}u_n + \frac{a}{n}u_{a+b+1} + \frac{b}{n}u_{a+b} \right\} \\ h^{-1}(h(\epsilon^a j \sigma_1) \cap \epsilon^a j L_1^*) &= \left\{ \frac{a}{n}\epsilon jt + \frac{b}{n}jt + \frac{a}{n}\epsilon^{a+b}t + \frac{b}{n}\epsilon^{a+b-1}t \right\} = \{w^*\}. \end{aligned}$$

(B) From $\sigma = [\epsilon^{a+b-1}t, \epsilon^{a+b}t; jt, \epsilon jt] = \epsilon^{-2a-b} \sigma_2$, we conclude that

$$\begin{aligned} h(\sigma_2) \cap L_1^* &= \left\{ \frac{a}{n}u_a + \frac{b}{n}u_{a+1} + \frac{a}{n}u_{2a+b} + \frac{b}{n}u_{2a+b+1} \right\} \\ h(\epsilon^{-2a-b} \sigma_2) \cap \epsilon^{-2a-b} L_1^* &= \left\{ \frac{a}{n}u_{a+b} + \frac{b}{n}u_{a+b+1} + \frac{a}{n}u_n + \frac{b}{n}u_{a+b} \right\} \\ h^{-1}(h(\epsilon^{-2a-b} \sigma_2) \cap \epsilon^{-2a-b} L_1^*) &= \left\{ \frac{a}{n}\epsilon^{a+b-1}t + \frac{b}{n}\epsilon^{a+b}t + \frac{a}{n}jt + \frac{b}{n}\epsilon jt \right\} = \{v^*\}; \end{aligned}$$

(C) From $\sigma = [\epsilon^{a+b-1}t, \epsilon^{a+b}t; jt, \epsilon jt] = \theta_1$, it easily follows

$$\begin{aligned} h(\theta_1) \cap L_1^* &= \left\{ \frac{b}{n}u_{a+b} + \frac{a}{n}u_{a+b+1} + \frac{b}{n}u_n + \frac{a}{n}u_1 \right\} \\ h^{-1}(h(\theta_1) \cap L_1^*) &= \left\{ \frac{b}{n}\epsilon^{a+b-1}t + \frac{a}{n}\epsilon^{a+b}t + \frac{b}{n}jt + \frac{a}{n}\epsilon jt \right\} = \{w^*\}. \end{aligned}$$

Conclusion 14 *Therefore,*

$$\text{card}(h(e) \cap (\cup \mathcal{A}(J, \alpha))) = \text{card}(h^{-1}(h(e) \cap (\cup \mathcal{A}(J, \alpha)))) = 2$$

and

$$h^{-1}(h(e) \cap (\cup \mathcal{A}(J, \alpha))) = \{v^*, w^*\}.$$

4.5 Step 4

Before analyzing the obstruction cocycle, let us observe that

$$\begin{aligned} h(v^*) = \epsilon^b \cdot v \in & \epsilon^{-2a-b}(L_1^* \cap L_2^*) \cap \epsilon^{-a}(L_1^* \cap L_2^*) \cap \epsilon j \epsilon^{-a-b+1}(L_1^* \cap L_2^*) \cap \epsilon j \epsilon(L_1^* \cap L_2^*) = \\ & \epsilon^b(L_1^* \cap L_2^*) \cap \epsilon^{2b+a}(L_1^* \cap L_2^*) \cap \epsilon^{a+b}j(L_1^* \cap L_2^*) \cap j(L_1^* \cap L_2^*) = \\ & \epsilon^b(L_1^* \cap \epsilon^{b+a}L_1^* \cap \epsilon^a j L_1^* \cap \epsilon^{2a+b}j L_1^* \cap L_2^*) \end{aligned}$$

and similarly

$$\begin{aligned} h(w^*) = w \in & (L_1^* \cap L_2^*) \cap \epsilon^{-a-b}(L_1^* \cap L_2^*) \cap \epsilon j \epsilon^{-a+1}(L_1^* \cap L_2^*) \cap \epsilon j \epsilon^{-2a-b+1}(L_1^* \cap L_2^*) = \\ & (L_1^* \cap L_2^*) \cap \epsilon^{a+b}(L_1^* \cap L_2^*) \cap \epsilon^a j(L_1^* \cap L_2^*) \cap \epsilon^{2a+b}j(L_1^* \cap L_2^*) = \\ & L_1^* \cap \epsilon^{b+a}L_1^* \cap \epsilon^a j L_1^* \cap \epsilon^{2a+b}j L_1^* \cap L_2^* \end{aligned}$$

Then the equality 7 applied in this situation implies that the obstruction cocycle is

$$c_{\mathbb{Q}_{4n}}(h)(e) = \|h(v^*)\| + \|h(w^*)\|$$

where $\tau_1, \tau_2 \in \{+1, -1\}$. In this situation classes $\|h(v^*)\|$ and $\|h(w^*)\|$ are broken point classes determined by the h embedding of the boundary of the simplex σ , i.e. by the linear subspace $\text{span}\{u_{a+b}, u_{a+b+1}, u_n, u_1\}$.

Since, $h(v^*) = \epsilon^b \cdot v$ and $h(w^*) = w$ the obstruction cocycle is

$$c_{\mathbb{Q}_{4n}}(h)(e) = \epsilon^b \cdot \|v\| + \|w\|.$$

where the first broken point class is determined by the linear span of the simplex

$$\epsilon^{-b}[u_{a+b}, u_{a+b+1}, u_n, u_1] = [u_a, u_{a+1}, u_{2a+b}, u_{2a+b+1}].$$

Also, observe that the equality

$$\epsilon^a j \cdot v = \epsilon^a j \cdot \left(\frac{a}{n} u_a + \frac{b}{n} u_{a+1} + \frac{a}{n} u_{2a+b} + \frac{b}{n} u_{2a+b+1} \right) = \frac{a}{n} u_1 + \frac{b}{n} u_n + \frac{a}{n} u_{a+b+1} + \frac{b}{n} u_{a+b} = w$$

and the fact that the permutation (4321) is even implies

$$\|w\| = \epsilon^a j \cdot \|v\|.$$

Conclusion 15 *The cohomology class of the obstruction cocycle, as we have seen, lives in the group of coinvariants $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}}$. Thus, instead of the original obstruction cocycle $c_{\mathbb{Q}_{4n}}(h)(e) = \epsilon^b \cdot \|v\| + \|w\| = (\epsilon^b + \epsilon^a j) \cdot \|v\|$ we can analyze the cocycle*

$$c' = 2 \|v\|$$

where the broken point class $\|v\|$ is determined by the linead span of the simplex

$$[u_a, u_{a+1}, u_{2a+b}, u_{2a+b+1}].$$

4.6 Step 5

Now we will find the complete second homology $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})$ via the isomorphism (4). In order to do so we describe the first two lower levels of the Hasse diagram of the intersection poset $P(\alpha)$ of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$. With a little linear algebra and mentioned properties of the added arrangement \mathcal{J} , it can be seen that the $n-4$ and $n-5$ levels of the Hasse diagram of the intersection poset $P(\alpha)$ are like in the figure 7.

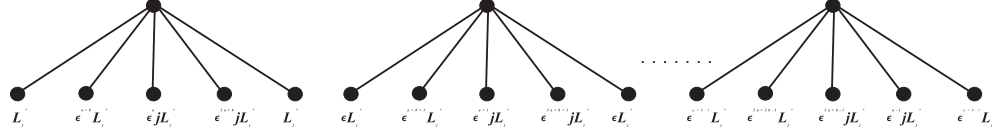


Figure 7: The $n-4$ and $n-5$ level of the intersection poset.

- (A) $L_1^* \cap \epsilon^{b+a} L_1^* \cap \epsilon^a j L_1^* \cap \epsilon^{2a+b} j L_1^* \cap L_2^*$ is a linear space of dimension $n-5$
- (B) L_1^* is a half-subspace of dimension $n-4$
- (C) L_2^* is a linear space of dimension $n-4$

we conclude that

$$\begin{aligned} H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z}) &\cong \mathbb{Z}^{a+b} \oplus \mathbb{Z}^{4(a+b)} \oplus \bigoplus_{d=0}^{n-6} \text{Hom} \left(\bigoplus_{\dim V=d} H_{n-4}(\Delta(P_{<V}) * \hat{V}); \mathbb{Z} \right) \\ &\cong \mathbb{Z}^{a+b} \oplus \mathbb{Z}^{4(a+b)} \oplus \bigoplus_{d=0}^{n-6} \text{Hom} \left(\bigoplus_{\dim V=d} \tilde{H}_{n-5-\dim V}(\Delta(P_{<V})); \mathbb{Z} \right). \end{aligned}$$

Lemma 16 $(\forall V \in P(\alpha)) \dim V \leq n-6 \implies \tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = 0.$

Proof. For every element $W \in P(\alpha)_{<V}$ such that $\dim W = n-4$ there exists a unique element $U_W \in P(\alpha)_{<V}$ with the property $\dim U_W = n-5$ and $W < U_W$. There is a monotone map $f : P(\alpha)_{<V} \rightarrow P(\alpha)_{<V} - \{U \mid \dim U = n-4\}$ defined by

$$U \longmapsto \begin{cases} U, & \text{for } \dim U \leq n-5 \\ U_W & \text{for } \dim U = n-4 \end{cases}$$

which satisfies conditions of the Quillen fiber lemma. Thus, f induces a homotopy equivalence and so

$$\tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V})) = \tilde{H}_{n-5-\dim V}(\Delta(P(\alpha)_{<V} - \{U \mid \dim U = n-4\})) = 0.$$

because $\dim \Delta(P(\alpha)_{<V} - \{U \mid \dim U = n-4\}) < n-5-\dim V$. ■

Conclusion 17 *The previous lemma and equality implies that*

$$H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z}) \cong H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z}) \cong \mathbb{Z}^{a+b} \oplus \mathbb{Z}^{4(a+b)}. \quad (8)$$

4.7 Step 6

We compute \mathbb{Q}_{4n} coinvariants by working in the \mathbb{Q}_{4n} -module $H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ with the modified action. Since action respects dimensional decomposition (4) we analyze $n-4$ and $n-5$ dimension cases separately.

(A) Let $l \in H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ be the element which corresponds to the compactification of the subspace L_2^* . Then $l, \epsilon l, \dots, \epsilon^{a+b-1}l$ is a base of the first factor in the equality (8). There are to set identities which should be considered.

The identity $L_2^* = \epsilon^{a+b} L_2^*$ produces equality $l = \det(\epsilon^{a+b}) \epsilon^{a+b} \cdot l = (-1)^{(n+1)(a+b)} \epsilon^{a+b} \cdot l$ in homology. Indeed, like in examples 11 and 12, we look at the orthogonal complement of L_2^* and see that ϵ^{a+b} acts on a basis as the even permutation (3412). Then coinvariant calculation with modified action implies a trivial relation,

$$l \sim \epsilon^{a+b} * l = \det(\epsilon^{a+b}) \epsilon^{a+b} \cdot l = \det(\epsilon^{a+b})^2 l.$$

The second identity $L_2^* = \epsilon^{-b} j L_2^*$ produces equality $l = (-1) \det(\epsilon^{-b} j) \epsilon^{-b} j \cdot l$ in homology. Like we mentioned, this is the consequence of the fact that $\epsilon^{-b} j$ acts on a base of $(L_2^*)^\perp$ as the odd permutation (3214). Then,

$$l \sim \epsilon^{-b} j * l = \det(\epsilon^{-b} j) \epsilon^{-b} j \cdot l = -\det(\epsilon^{-b} j)^2 \cdot l = -l.$$

Thus, the first factor in the equality (8) reduces in the coinvariants to a single \mathbb{Z}_2 .

(B) Since we want to find the coinvariants we can concentrate on the "generating" part of the Hasse diagram and its inner symmetries (figure 8)

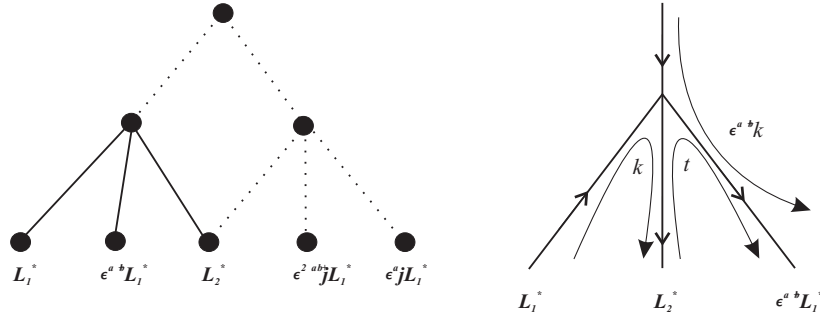


Figure 8: Generating part of the Hasse diagram for coinvariants.

Let k and h be the elements of $H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ which correspond to the compactifications of the intersections $L_1^* \cap L_2^*$ and $\epsilon^{a+b} L_1^* \cap L_1^*$. Then the orbits of these two elements form a base of the second factor in the equality (8). Let $t \in H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ also denotes the element which corresponds to the compactification of the intersection $\epsilon^{a+b} L_1^* \cap L_2^*$ with the same generic point in L_2^* as in the representation of the k . Here we have to be very careful, because ϵ^{a+b} interchanges the halfspaces of L_2^* generated by the hyperplane H_1 . Moreover Thus, like the figure 8 indicates

$$\epsilon^{a+b} \cdot k = \det(\epsilon^{a+b})(l + t) \quad (9)$$

where $l \in H_{n-4}(\cup \widehat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ (as in part (A)) is homology class corresponding to \widehat{L}_2^* .

On the other hand $k + t = h$. Since the sphere $L_1^* \cap \widehat{\epsilon^{a+b} L_1^*} * S^0$ is invariant of the element ϵ^{a+b} , there is a possible equality $h = \pm \epsilon^{a+b} \cdot h$. Let us fix the basis $\{e_1 + \dots + e_a, e_{a+1} + \dots + e_{a+b}, e_{a+b+1} + \dots + e_{2a+b}, e_{2a+b+1} + \dots + e_n, (a+b)(e_a - e_{2a+b} + e_1 - e_{a+b+1}) + e_{a+1} - e_{2a+b+1} + e_n - e_{a+b}\}$ of the orthogonal complement of $L_1^* \cap \epsilon^{a+b} L_1^*$. Then the matrix of ϵ^{a+b} on this basis is

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and $\det M = -1$. This produces the relation $h = -\det(\epsilon^{a+b})\epsilon^{a+b} \cdot h$ in homology. Thus,

$$h \sim \epsilon^{a+b} * h = \det(\epsilon^{a+b})\epsilon^{a+b} \cdot h = -\det(\epsilon^{a+b})^2 h = -h. \quad (10)$$

Since, $\epsilon^{a+b} \cdot k - \det(\epsilon^{a+b})t = \det(\epsilon^{a+b})l$ and $h = k + t$, the relations 9 and 10 imply that

$$l \sim \det(\epsilon^{a+b})l = \det(\epsilon^{a+b})(\epsilon^{a+b} \cdot k - \det(\epsilon^{a+b})t) = \epsilon^{a+b} * k - t \sim k - t \text{ and } k + t \sim -k - t.$$

Actually, we conclude that $4k = 0$, or the second factor produces one \mathbb{Z}_4 .

Conclusion 18 $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}} \cong H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

4.8 Step 7

Now we are ready to find the image of the obstruction cocycle $c' = 2\|v\|$ via the isomorphism $\vartheta : H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z}) \rightarrow \text{Hom}(\widehat{H_{n-4}(\cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})}, \mathbb{Z})$. Actually we have to compute (from (5)) the linking numbers of the sphere / boundary of the simplex $S = \partial[u_a, u_{a+1}, u_{2a+b}, u_{2a+b+1}]$ (which represents the broken point class $\|v\|$) and spheres which represents generators of the $H_n(\cup \widehat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$. Since simplex $[u_a, u_{a+1}, u_{2a+b}, u_{2a+b+1}]$ intersects only $I = L_1^* \cap \epsilon^{b+a} L_1^* \cap \epsilon^a j L_1^* \cap \epsilon^{2a+b} j L_1^* \cap L_2^*$, the homology elements which may have non-zero values are those from the local picture $H_n(\widehat{I} * [5]; \mathbb{Z})$. Since the coinvariant class of l lives in \mathbb{Z}_2 and our obstruction cocycle is $2 \cdot (\text{something})$ there is no need to compute the image of l either.

By moving simplex $[u_a, u_{a+1}, u_{2a+b}, u_{2a+b+1}]$ and analyzing its intersection with subspaces $L_1^*, \epsilon^{b+a} L_1^*, \epsilon^a j L_1^*, \epsilon^{2a+b} j L_1^*, L_2^*$ we decompose the broken point class $\|v\|$ into a sum of ordinary point classes.

Theorem 19 *Let $v_1 \in L_1^*$, $v_2 \in \epsilon^{b+a} L_1^*$, $v_3 \in \epsilon^a j L_1^*$, $v_4 \in \epsilon^{2a+b} j L_1^*$ and $y_1, y_2 \in L_2^*$ are arbitrary elements with the property that they do not belong to any other element of the arrangement $\mathcal{A}(\mathcal{J}, \alpha)$. The request for y_1 and y_2 is that they are in different connecting components of $L_2^* \setminus I$. Then our broken point class $\|v\|$ can be presented as the sum of the point classes*

$$\|v\| = \tau_1 \|v_1\| + \tau_2 \|v_3\| + \tau_3 \|y_1\| = \mu_1 \|v_2\| + \mu_2 \|v_4\| + \mu_3 \|y_2\| \quad (11)$$

where τ_i and μ_i are appropriate signs $\{+1, -1\}$.

Proof. The idea of the proof is very simple. *Move our filled sphere / simplex a little and look where it hits our arrangement and we are done.* The reason for this to work is in the definition of the linking - intersection number and the definition of (broken) point classes. Let us recall that the linking /intersection number is "very" invariant under small movement of the sphere - disc. Thus if you want to check whether you calculated the intersection number properly you move your sphere - disc and see what happens. Since we are in linear / convex situation, besides the linking - intersection numbers (if correctly defined) are +1,-1 or 0, it is enough to consider small translations of a sphere - disc. Thus, let us move our simplex σ by the generic "small" vector $s = \sum_{i=1}^n \xi_i e_i$. Because of complementary dimension affine space $\text{span}\{u_{a+b}, u_{a+b+1}, u_n, u_1\} + s$ must hit all linear spans of L_1^* , $\epsilon^{a+b} L_1^*$, $\epsilon^a j L_1^*$, $\epsilon^{2a+b} j L_1^*$ and L_2^* . Let us denote these intersection points by

$$\begin{aligned} v_1 &\in \text{span}(L_1^*) \cap (\sigma + s), \quad v_2 \in \text{span}(\epsilon^{a+b} L_1^*) \cap (\sigma + s), \quad v_3 \in \text{span}(\epsilon^a j L_1^*) \cap (\sigma + s), \\ v_4 &\in \text{span}(\epsilon^{2a+b} j L_1^*) \cap (\sigma + s), \quad w \in \text{span}(L_2^*) \cap (\sigma + s). \end{aligned}$$

If the translation is small enough these points will remain in the interior of the simplex $\sigma + s$. Now the tiresome, but necessary part. The points are given by

$$\begin{aligned} v_1 &= \frac{\alpha_1}{n} e_a + \frac{\beta_1}{n} e_{a+1} + \frac{\gamma_1}{n} e_{2a+b} + \frac{\delta_1}{n} e_{2a+b+1} - \frac{1}{n} \sum_{i=1}^n e_i + \sum_{i=1}^n \xi_i e_i, \\ \alpha_1 &= a - n \sum_1^a \xi_i, \quad \gamma_1 = a + b - n \sum_{a+1}^{2a+b} \xi_i - \beta_1, \quad \delta_1 = b - n \sum_{2a+b+1}^n \xi_i, \\ \beta_1 &= \frac{-n}{a+b+1} \left(\sum_{2a+b+2}^n \xi_i + \xi_n - \xi_{a+b} + \xi_{a+1} - (a+b) \left(\sum_2^{a-1} \xi_i - \sum_{a+1}^{2a+b-1} \xi_i + \xi_{a+b+1} \right) \right) + b \\ v_2 &= \frac{\alpha_2}{n} e_a + \frac{\beta_2}{n} e_{a+1} + \frac{\gamma_2}{n} e_{2a+b} + \frac{\delta_2}{n} e_{2a+b+1} - \frac{1}{n} \sum_{i=1}^n e_i + \sum_{i=1}^n \xi_i e_i, \\ \alpha_2 &= a + b - n \sum_{2a+b+1}^n \xi_i - n \sum_{i=1}^a \xi_i - \delta_2, \quad \beta_2 = b - n \sum_{a+1}^{a+b} \xi_i, \quad \gamma_2 = a - n \sum_{a+b+1}^{2a+b} \xi_i, \\ \delta_2 &= \frac{-n}{a+b+1} \left(\sum_{a+2}^{a+b} \xi_i - \xi_n + \xi_{a+b} + \xi_{2a+b+1} + (a+b) \left(\sum_{2a+b+1}^n \xi_i + \sum_2^{a-1} \xi_i - \sum_{a+b+2}^{2a+b-1} \xi_i \right) \right) + b; \\ v_3 &= \frac{\alpha_3}{n} e_a + \frac{\beta_3}{n} e_{a+1} + \frac{\gamma_3}{n} e_{2a+b} + \frac{\delta_3}{n} e_{2a+b+1} - \frac{1}{n} \sum_{i=1}^n e_i + \sum_{i=1}^n \xi_i e_i, \\ \alpha_3 &= a - n \sum_1^a \xi_i, \quad \beta_3 = b - n \sum_{a+1}^{a+b} \xi_i, \quad \gamma_3 = a + b - n \sum_{a+b+1}^n \xi_i - \delta_3 \\ \delta_3 &= \frac{-n}{a+b-1} \left((a+b) \left(\sum_{a+b+2}^n \xi_i - \sum_2^{a-1} \xi_i - \xi_{2a+b} \right) - \sum_{a+2}^{a+b} \xi_i - \xi_{2a+b+1} - \xi_{a+b} + \xi_n \right) + b; \end{aligned}$$

$$\begin{aligned}
v_4 &= \frac{\alpha_4}{n}e_a + \frac{\beta_4}{n}e_{a+1} + \frac{\gamma_4}{n}e_{2a+b} + \frac{\delta_4}{n}e_{2a+b+1} - \frac{1}{n}\sum_{i=1}^n e_i + \sum_{i=1}^n \xi_i e_i, \\
\alpha_4 &= a+b-n\sum_1^{a+b} \xi_i - \beta_4, \delta_4 = b-n\sum_{2a+b+1}^n \xi_i, \gamma_4 = a-n\sum_{a+b+1}^{2a+b} \xi_i \\
\beta_4 &= \frac{n}{a+b-1} \left((a+b) \left(\sum_{a+b+1}^{2a+b} \xi_i - \sum_2^{a+b} \xi_i + \xi_a \right) - \sum_{2a+b+1}^{n-1} \xi_i - \xi_{2a+b+1} + \xi_{a+1} - \xi_{a+b} \right) + b; \\
w_4 &= \frac{\alpha_5}{n}e_a + \frac{\beta_5}{n}e_{a+1} + \frac{\gamma_5}{n}e_{2a+b} + \frac{\delta_5}{n}e_{2a+b+1} - \frac{1}{n}\sum_{i=1}^n e_i + \sum_{i=1}^n \xi_i e_i, \\
\alpha_5 &= a-n\sum_1^a \xi_i, \beta_5 = b-n\sum_{a+1}^{a+b} \xi_i, \gamma_5 = a-n\sum_{a+b+1}^{2a+b} \xi_i, \delta_5 = b-n\sum_{2a+b+1}^n \xi_i.
\end{aligned}$$

Now we will prove the following equivalences

$$\begin{aligned}
&(v_1 \in K^+ \Leftrightarrow v_2 \in K^+) \vee (v_1 \in K^- \Leftrightarrow v_2 \in K^-) \\
&(v_3 \in \epsilon^a j K^+ \Leftrightarrow v_4 \in \epsilon^a j K^+) \vee (v_3 \in \epsilon^a j K^- \Leftrightarrow v_4 \in \epsilon^a j K^-)
\end{aligned}$$

which have the following consequence

$$(v_1 \in L_1^* \Leftrightarrow v_2 \notin \epsilon^{a+b} L_1^*) \text{ and } (v_3 \in \epsilon^a j L_1^* \Leftrightarrow v_4 \notin \epsilon^{2a+b} j L_1^*) \quad (12)$$

and therefore the equality (11).stands.

In order to prove equivalences (12) we evaluate v_1, v_2, v_3 and v_4 on linear forms $x_1 + \dots + x_{a+b}, x_{a+b+1} + \dots + x_n, x_{2a+b+1} + \dots + x_n + x_1 + \dots + x_a$ and $x_{a+1} + \dots + x_{2a+b}$, respectively. It can be calculated that

$$\begin{aligned}
x_1 + \dots + x_{a+b} \mid_{v_1} &= x_{a+1} + \dots + x_{a+b} \mid_{v_1} = \frac{\beta_1}{n} - \frac{b}{n} + \sum_{a+1}^{a+b} \xi_i \\
&= \frac{a+b}{a+b+1} \sum_2^{a-1} \xi_i + \frac{1}{a+b+1} \sum_{a+2}^{a+b-1} \xi_i + \frac{2}{a+b+1} \xi_{a+b} \\
&\quad - \frac{a+b}{a+b+1} \sum_{a+b+2}^{2a+b-1} \xi_i - \frac{1}{a+b+1} \sum_{2a+b+2}^{n-1} \xi_i - \frac{2}{a+b+1} \xi_n \\
x_{a+b+1} + \dots + x_n \mid_{v_2} &= x_{2a+b+1} + \dots + x_n \mid_{v_2} = \frac{\delta_2}{n} - \frac{b}{n} + \sum_{2a+b+1}^n \xi_i \\
&= -\frac{a+b}{a+b+1} \sum_{i=2}^{a-1} \xi_i - \frac{1}{a+b+1} \sum_{a+2}^{a+b-1} \xi_i - \frac{2}{a+b+1} \xi_{a+b} \\
&\quad + \frac{a+b}{a+b+1} \sum_{i=a+b+2}^{2a+b-1} \xi_i + \frac{1}{a+b+1} \sum_{2a+b+2}^{n-1} \xi_i + \frac{2}{a+b+1} \xi_n \\
x_{2a+b+1} + \dots + x_n + x_1 + \dots + x_a \mid_{v_3} &= x_{2a+b+1} + \dots + x_n \mid_{v_2} = \frac{\delta_3}{n} - \frac{b}{n} + \sum_{2a+b+1}^n \xi_i \\
&= \frac{a+b}{a+b-1} \sum_2^{a-1} \xi_i + \frac{1}{a+b-1} \sum_{a+2}^{a+b-1} \xi_i + \frac{2}{a+b-1} \xi_{a+b} \\
&\quad - \frac{a+b}{a+b-1} \sum_{a+b+2}^{2a+b-1} \xi_i - \frac{1}{a+b-1} \sum_{2a+b+2}^{n-1} \xi_i - \frac{2}{a+b-1} \xi_n \\
x_{a+1} + \dots + x_{2a+b} \mid_{v_4} &= x_{a+1} + \dots + x_{a+b} \mid_{v_4} = \frac{\beta_4}{n} - \frac{b}{n} + \sum_{a+1}^{a+b} \xi_i \\
&= -\frac{a+b}{a+b-1} \sum_2^{a-1} \xi_i - \frac{1}{a+b-1} \sum_{a+2}^{a+b-1} \xi_i - \frac{2}{a+b-1} \xi_{a+b} \\
&\quad + \frac{a+b}{a+b-1} \sum_{a+b+2}^{2a+b-1} \xi_i + \frac{1}{a+b-1} \sum_{2a+b+2}^{n-1} \xi_i + \frac{2}{a+b-1} \xi_n
\end{aligned}$$

Thus, we have proved

$$\begin{aligned}
(a+b+1)(x_1+\dots+x_{a+b} \mid_{v_1}) &= -(a+b+1)(x_{a+b+1}+\dots+x_n \mid_{v_2}) \\
&= (a+b-1)(x_{2a+b+1}+\dots+x_n+x_1+\dots+x_a \mid_{v_3}) \\
&= -(a+b+1)(x_{a+1}+\dots+x_{2a+b} \mid_{v_4})
\end{aligned}$$

which implies the equivalence (12).

It looks like we forgot about the space L_2^* and point w . The reason is that when ever we move the simplex σ it will hit the space L_2^* . In respect of the direction we move σ the point w will be in one or in another connecting component of $L_2^* - I$. Thus, the broken point class $\|w\|$ appears in both equalities. Also, observe that we do not worry about the signs τ_i and μ_i , they will not play any role in our calculation.

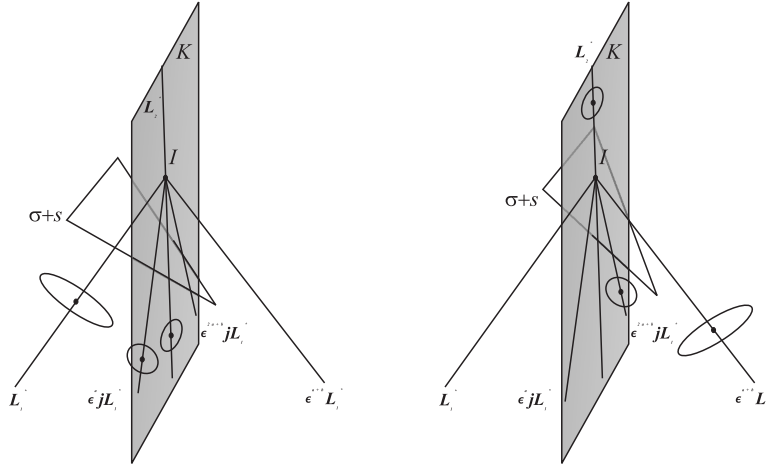


Figure 9: Generating part of the Hasse diagram for coinvariants.

■

Conclusion 20 *Let us observe that $\|v_1\| = \pm \epsilon^a j \|v_3\|$ and $\|v_2\| = \pm \epsilon^a j \|v_4\|$. Thus, since the class of our obstruction cocycle*

$$c' = 2 \|v\| = 2 (\tau_1 \|v_1\| + \tau_2 \|v_3\| + \tau_3 \|y_1\|) = 2 (\mu_1 \|v_2\| + \mu_2 \|v_4\| + \mu_3 \|y_2\|)$$

lives in coinvariants $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ we can shift our attention to the cohomological cocycle

$$c'' = 2\tau_3 \|y_1\|.$$

4.9 Step 8

Before proving that the cohomology class of the obstruction cocycle $c'' = 2\tau_3 \|y_1\|$ is not zero, let us recall the nature of the Poincaré duality isomorphism $\vartheta : H_{m-1}(\mathbb{R}^{n+m} \setminus \cup \mathcal{A}; \mathbb{Z}) \rightarrow$

$\text{Hom}(H_n(\cup \hat{\mathcal{A}}; \mathbb{Z}), \mathbb{Z})$. As we have seen, for $t \in H_n(\cup \hat{\mathcal{A}}; \mathbb{Z})$ represented by a submanifold T in $\cup \hat{\mathcal{A}}$, the image $\vartheta(\|x\|)(t)$ of t is given by the linking number

$$\vartheta(\|x\|)(t) = \text{link}(S, T)..$$

Theorem 21 *As we have already denoted, let $l, k, h \in H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ be elements corresponding to spheres \widehat{L}_2^* , $\widehat{L_1^* \cap L_2^*} * S^*$, $\epsilon^{a+b} \widehat{L_1^* \cap L_1^*} * S^*$. Then there exists a basis of $H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ which contains l, k , and h such that*

$$\vartheta(\|y_1\|)(l) = \pm 1, \vartheta(\|y_1\|)(k) = \pm 1, \vartheta(\|y_1\|)(h) = 0 \quad (13)$$

and for every other base element $d \in H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$, $\vartheta(\|y_1\|)(d) = 0$. Thus,

(1) $\vartheta(\|y_1\|) = \mu_1 l + \mu_2 k \in H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ where $\mu_i \in \{+1, -1\}$;

(2) $c'' = 2k \neq 0$ in coinvariants $H_2(W_n \setminus \cup \mathcal{A}(\mathcal{J}, \alpha); \mathbb{Z})_{\mathbb{Q}_{4n}} \cong H_n(\cup \hat{\mathcal{A}}; \mathbb{Z})_{\mathbb{Q}_{4n}}$.

We abuse notation in (2) and (3) using the fact that $H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ is free (Hom can be deleted) and omitting the notation of the cohomology class to simplify notation.

Proof. The proof can be read from the figure 9. For example, the required basis of $H_{n-4}(\cup \hat{\mathcal{A}}(\mathcal{J}, \alpha); \mathbb{Z})$ is composed of

(A) the orbit of l ,

(B) $k, h, \epsilon^{a+b}h$ and element f corresponding to the sphere $\epsilon^{a+b} \widehat{L_1^* \cap \epsilon^a j L_1^*}$,

(C) anything you want outside the I "neighbourhood", or precisely $\epsilon k, \epsilon h, \epsilon^{a+b+1}h, \epsilon f, \dots, \epsilon^{a+b-1}k, \epsilon^{a+b-1}h, \epsilon^{n-1}h, \epsilon^{a+b-1}f$.

This basis satisfies the property (13). Thus (1) and (2) follow directly with the knowing that $4k = 0$ in coinvariants. ■

Conclusion 22 *Since we proved that the cohomology class of the obstruction cocycle is not zero we are ready to sum up (remember Propositions 5 and 4):*

$$\begin{aligned} \text{There is no } \mathbb{Q}_{4n}\text{-map } S^3 \rightarrow \cup \mathcal{A}(\mathcal{J}, \alpha) &\implies \text{There is no } \mathbb{Q}_{4n}\text{-map } S^3 \rightarrow \cup \mathcal{A} \\ &\iff \text{There is no } \mathbb{D}_{2n}\text{-map } V_2(\mathbb{R}^3) \rightarrow \cup \mathcal{A} \\ &\implies \text{There is an } (\frac{a}{n}, \frac{a+b}{n}, \frac{b}{n})\text{-partition.} \end{aligned}$$

Once more, we have just proved that for every $a, b > 1$ and every two measures μ and ν on S^2 , there exists an $(\frac{a}{n}, \frac{a+b}{n}, \frac{b}{n})$ -partition $(x; l_1, l_2, l_3)$ of measures μ and ν .

4.10 The limit argument

The reasons for assuming nice properties for our measures will finally pop up. Let $S \subseteq \mathbb{R}_{>0}^3$ be the space of all 3-fan partitions, i.e. the space of all triples $\alpha = (a, b, c)$, $a + b + c = 1$ such that there exists an α -partition of measures μ and ν by a 3-fan. Since our measures μ and ν are proper Borel probability measures, the space S is a closed subset of $\mathbb{R}_{>0}^3$. All we did so far is proving the following inclusion

$$\{(\frac{a}{n}, \frac{a+b}{n}, \frac{b}{n}) \in \mathbb{Q}_{>0}^3 \mid 2a + 2b = n, a, b \in \mathbb{Z}\} \subseteq S.$$

The fact that S is closed implies the inclusion

$$\{(a, b, c) \in \mathbb{R}_{>0}^3 \mid a + b + c = 1\} = \text{cl}\{(\frac{a}{n}, \frac{a+b}{n}, \frac{b}{n}) \mid 2a + 2b = n, a, b \in \mathbb{Z}\} \subseteq S.$$

and therefore we are finally done.

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